Quantum Field Theory

Ling-Fong Li

National Center for Theoretical Science

QCD

Quark Model

Isospin symmetry

To a good approximation, nuclear force is independent of the electromagnetic charge carried by the nucleons — charge independence. To implement this, we say strong interaction has an $SU(2)$ symmetry $n \leftrightarrow p$. These $SU(2)$ generators T_1, T_2, T_3 satisfy commutation ,

$$[T_i, T_j] = i\epsilon_{ijk} T_k$$

Acting on n or p

$$T_3|p\rangle = \frac{1}{2}|p\rangle, \quad T_3|n\rangle = -\frac{1}{2}|n\rangle, \quad T_+|n\rangle = |p\rangle, \quad T_-|p\rangle = |n\rangle \ \cdots$$

This means n and p form a doublet under isospin transformation. Isospin invariance simply means that

$$[T_i, H_s] = 0$$

where H_s is strong interation Hamiltonian.

We can extend isospin assignments to other hadrons . For example we get the following isospin multiplets,

$$(\pi^+, \pi^0, \pi^-) \quad I = 1, \qquad (K^+, K^0), \ (\bar{K^0}, K^-) \quad I = \frac{1}{2} \ , \qquad \eta \ , \quad I = 0$$

$$(\Sigma^+, \Sigma^0, \Sigma^-) \quad I = 1, \qquad (\Xi^0, \Xi^-), \quad I = \frac{1}{2}\ , \qquad \Lambda,\ I = 0$$

$$(\rho^+, \rho^0, \rho^-) \quad I = 1,\ (K^{+*}, K^{0*}), (\bar{K^0*}, K^{*-}) \quad I = \frac{1}{2} \quad \cdots$$

If isospin symmetry were exact, then all particles in multiplets have same masses, which is not the case in nature. But the mass difference within the isospin multiplets seems to be quite small.

$$\frac{m_n - m_p}{m_n + m_p} \sim 0.7 \times 10^{-3}, \qquad \frac{m_{\pi+} - m_{\pi 0}}{m_{\pi+} + m_{\pi 0}} \sim 1.7 \times 10^{-2} \quad \cdots$$

Thus isospin symmetry as approximate one and maybe it is good to few %.

SU(3) symmetry and Quark Model

When Λ and k particles were discovered, they were produced in pair (associated production) with longer life time. It was postulated that these new particles possessed a new additive quantum number, **strangeness** S , conserved by strong interaction but violated in decays,

$$S(\Lambda^0) = -1, \quad S(K^0) = 1 \quad \cdots$$

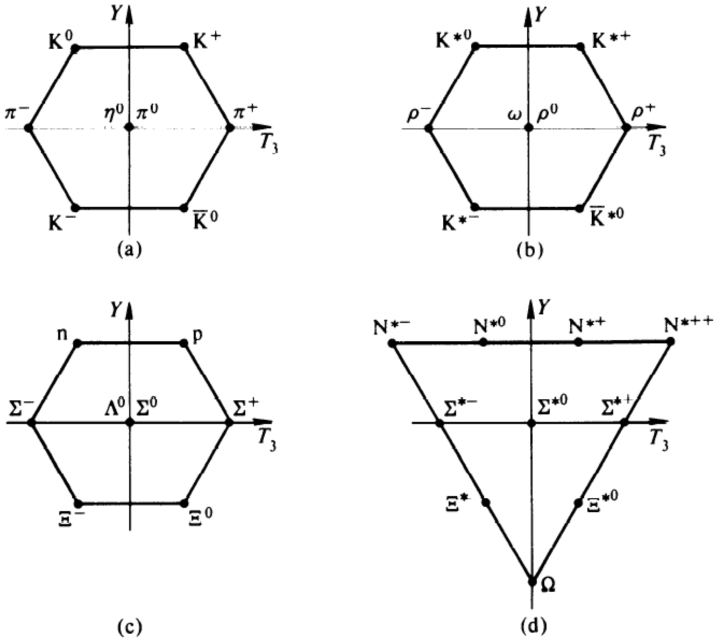
Extension to other hadrons, we can get a general relation,

$$Q = T_3 + \frac{Y}{2}$$

where $Y = B + S$ is called hyperchargee, and B is the baryon number. This is known as Gell-Mann-Nishijima relation.

Eight-fold way : Gell-Mann, Neeman

Group mesons or baryons with same spin and parity,



These are the same as irreducible representations of $SU(3)$ group. The spectra of hadrons show

Quark Model

One peculiar feature of eight fold way is that octet and decuplet are not the fundamental representation of $SU(3)$ group. In 1964, Gell-mann and Zweig independently propose the quark model: all hadrons are built out of spin $\frac{1}{2}$ quarks which transform as the fundamental representation of $SU(3)$,

$$q_i = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

with the quantum numbers

	Q	T	T_3	Y	S	B
u	$2/3$	$1/2$	$+1/2$	$1/3$	0	$1/3$
d	$-1/3$	$1/2$	$-1/2$	$1/3$	0	$1/3$
s	$-1/3$	0	0	$-2/3$	-1	$1/3$

In this scheme, mesons are $q\bar{q}$ bound states. For examples,

$$\pi^+ \sim \bar{d}u \quad \pi^0 \sim \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d). \quad \pi^- \sim \bar{u}d$$

$$K^+ \sim \bar{s}u \quad K^0 \sim \bar{s}d \ , \ K^- \bar{u}s. \quad \eta^0 \sim \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$$

and baryons are qqq bound states,

$$\begin{aligned} p &\sim uud \ , \ n \sim ddu \\ \Sigma^+ &\sim suu \ , \ \Sigma^0 \sim s\left(\frac{ud+du}{\sqrt{2}}\right) \ , \ \Sigma^- \sim sdd \\ \Xi^0 &\sim ssu \ , \ \Xi^- \sim ssd \ , \ \Lambda^0 \sim \frac{s(ud-du)}{\sqrt{2}}. \end{aligned}$$

Quantum numbers of the hadrons are all carried by the quarks. But we do not know the dynamics which bound the quarks into hadrons. Since quarks are the fundamental constituent of hadrons it is important to find these particles. But over the years none have been found.

Paradoxes of simple quark model

1.
   1. Quarks have fractional charges while all observed particles have integer charges. At least one of the quarks is stable. None has been found.
   2. Hadrons are exclusively made out $q\bar{q}, qqq$ bound states. In other word, $qq, qqqq$ states are absent.
   3. The quark content of the baryon N^{*++} is uuu. If we chose the spin state to be $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ then all quarks are in spin-up state˜ $\alpha_1\alpha_2\alpha_3$ is totally symmetric. If we assume that the ground state has $l = 0$, then spatical wave function is also symmetric. This will leads to violation of Pauli exclusion principle.

Color degree of freedom

One way to get out of these problems, is to introduce color degrees of freedom for each quark and postulates that only color singlets are physical observables. 3 colors are needed to get antisymmetric wave function for N^{*++} and remains a color singlet state. In other words each quark comes in 3 colors,

$$u_\alpha = (u_1, u_2, u_3) \quad , \quad d_\alpha = (d_1, d_2, d_3) \cdots$$

All hadrons form singlets under $SU(3)_{color}$ symmetry, e.g.

$$N^{*++} \sim u_\alpha(x_1)\alpha_\beta(x_2)u_\gamma(x_3)\varepsilon^{\alpha\beta\gamma}$$

Futhermore, color singlets can not be formed from the combination qq, $qqqq$ and they are absent from the observed specrum. Needless to say that a single quark is not observable.

Gell-Mann Okubo mass formula

Since $SU(3)$ is not an exact symmetry, we want to see whether we can understand the pattern of the $SU(3)$ breaking. Experimentally, $SU(2)$ seems to be a good symmetry, we will assume isospin symmetry to set $m_u = m_d$. We will assume that we can write the hadron masses as linear combinations of quark masses.

1. o^- mesons

$$\begin{aligned} m_\pi^2 &= (m_o + 2m_u) \\ m_k^2 &= m_o + m_u + m_s \\ m_\eta^2 &= m_o + \frac{2}{3}(m_u + 2m_s) \end{aligned}$$

Eliminate the quark masses we get

$$4m_k^2 = m_\pi^2 + 3m_\eta^2$$

This known as the Gell-Mann Okubo mass formula. Experimentally, we hav $LHS = 4m_k^2 \simeq 0.98(Gev)^2$ while $RHS = m_\pi^2 + 3m^2 \simeq 0.92(Gev)^2$ This seems to show that this formula works quite well.

2. $\frac{1}{2}^+$ baryon

$$\begin{aligned} m_N &= m_0 + 3m_u \\ m_\Sigma &= m_o + 2m_u + m_s \\ m_\Xi &= m_o + m_u + 2m_s \\ m_\Lambda &= m_o + 2m_u + m_s \end{aligned}$$

We get the mass relation

$$\frac{m_\Sigma + 3m_\Lambda}{2} = m_N + m_\Xi$$

Expermentally, $\frac{m_\Sigma + 3m_\Lambda}{2} \simeq 2.23\ Gev$, and $m_N + m_\Xi \simeq 2.25\ Gev$.

3. $\frac{3}{2}^+$ baryon

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_{N^*}$$

This is referred to as equal spacing rule. In fact when this relation is derived the particle Ω has not yet been found and this relation is used to predicted the mass of Ω and subsequent discovery gives a very strong support to $SU(3)$ symmetry.

$\omega - \phi$ mixing

For 1^- mesons, the situation seems to be different. If we make an analogy with o^- mesons, we would get Gell-Mann Okubo mass relation as,

$$3m_\omega^2 = 4m_{K^*}^2 - m_\rho^2$$

Using $m_{K^*} = 890\ Mev$, $m_\rho = 770\ Mev$ we get $m_\omega = 926.5\ Mev$ from this. But experimentally, $m_\omega = 783 Mev$ which is qiute far away. On the other hand there is a ϕ meson with mass $m_\phi = 1020\ Mev$ and has same $SU(2)$ quatntum number as ω. In principle, when $SU(3)$ symmetry is broken, $\omega - \phi$ mixing is possible. Suppose for some reason there is a significant $\omega - \phi$ mixing we want to see whether this can save the mass relation.

Denote the $SU(3)$ octet state by V_8 and singlet state by V_1

$$V_8 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s) \quad , \quad V_1 = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s)$$

Write the mass matrix as

$$M = \begin{pmatrix} m_{88}^2 & m_{18}^2 \\ m_{18}^2 & m_{11}^2 \end{pmatrix}$$

Assume that the octet mass is that predicted by Gell-Mann Okubo mass relation, i.e.

$$3{m^2}_{88} = 4m_{K^*}^2 - m_\rho^2$$

After diagonalizing the mass matrix M, we get

$$R^{+}MR = M_d = \begin{pmatrix} m_\omega^2 & 0 \\ 0 & m_\phi^2 \end{pmatrix}, \quad \text{with} \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Thus

$$\begin{aligned} \omega &= \cos\theta V_8 - \sin\theta V_1 \\ \phi &= \sin\theta V_8 + \cos\theta V_1 \end{aligned}$$

and

$$\sin\theta = \sqrt{\frac{(m_{88}^2 - m_\omega^2)}{(m_\phi^2 - m_\omega^2)}}$$

Using $m_{88} = 926.5 Mev$ from Gell-Mann Okubo mass formula, we get

$$\sin\theta = 0.76$$

This is very close to the ideal mixing $\sin\theta = \sqrt{\frac{2}{3}} = 0.81$ where mass eigenstates have a simple form,

$$\begin{aligned} \omega &= \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d) \\ \phi &= \bar{s}s \end{aligned}$$

This means that the physical ϕ meson is mostly made out of s quarks in this scheme.

1. **Zweig rule**
Since ω and ϕ have same quantum numbers under $SU(2)$, one expects they have similar decay widths. Experimentally, $\omega \to 3\pi$ mostly, but $\phi \to 3\pi$ is very suppressed relative to $\phi \to KK$ channel even though $\phi \to KK$ has very small phase space since $m_\phi = 1020$ *Mev* and $m_k \approx 494$ *Mev*.

$$B(\phi \to KK) \approx 85\% \quad , \quad B(\phi \to \pi\pi\pi) \sim 28\%$$

Quark diagrams
In term quarks contents, the decays of ϕ meson proceed as following diagrams

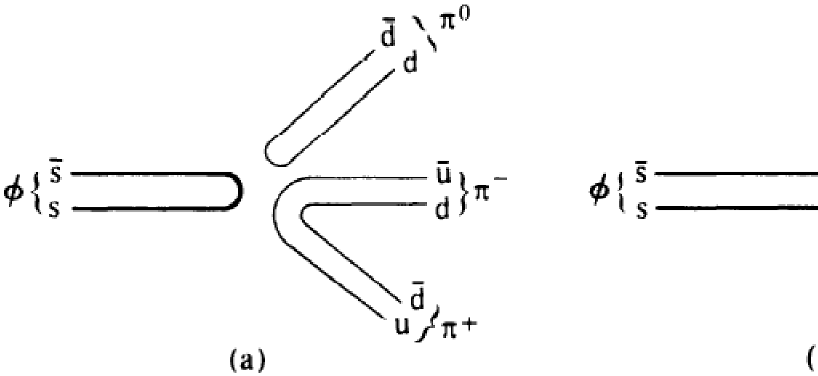


FIG. 4.10. φ decays: (a) disallowed; (b) allowed by t

Zweig rule postulates that processes involving quark-antiquark annihilation are highly suppressed for some reason. This explains why ϕ has a width $\Gamma_\phi \approx 4.26$ *Mev* smaaler than $\Gamma_\omega \approx 8.5$ *Mev*.

J/ψ and charm quark

In 1974 the $\psi/J(3100)$ particle was discovered with unusually narrow width, $\Gamma \sim 70$ *kev*, compared to $\Gamma_\rho \sim 150$ *Mev*, $\Gamma_\omega \sim 10$*Mev*.

Simple explanation, $\psi/J \sim \bar{c}c$ and is below the threshold of decaying into 2 mesons containing charm quark. It can only decay by $c\bar{c}$ annihilation in the initial state.By Zweig rule, these decays are highly suppressed and have very narrow width.

Asymptotic freedom

1. $\lambda\phi^4$ theory

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu)^2 - m^2\phi^2] - \frac{\lambda}{4!}$$

Effective coupling constant $\bar{\lambda}$ satisfies

$$\frac{d\bar{\lambda}}{dt} = \beta(\bar{\lambda}) \ , \quad \beta(\bar{x}) \approx \frac{3\lambda^2}{16\pi^2} + 0(\lambda^3)$$

It is not asymptotically free.
Generalization: $\lambda\phi^4 \longrightarrow \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l$, λ_{ijkl} is totally symmetric. It turns out that

$$\beta_{ijkl} = \frac{d\lambda_{ijkl}}{dt} = \frac{1}{16\pi^2}\left[\lambda_{ijmn}\lambda_{mnkl} + \lambda_{ikmn}\lambda_{mnjl} + \lambda_{ilmn}\lambda_{mnjk}\right]$$

Take the special case, $i = j = k = l$

$$\beta_{1111} = \frac{3}{16\pi^2}\lambda_{11mn}\lambda_{mn11} > 0$$

is not asymptotically free.

2. Yukawa interaction

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}[(\partial_\mu\phi)^2 - \mu^2\phi^2] - \lambda\phi^4 + f\bar{\psi}\psi\phi$$

The equations for effective coupling constant are

$$\beta_\lambda = \frac{d\lambda}{dt} = A\lambda^2 + B\lambda f^2 + Cf^4,\ A > 0$$

$$\beta_f = \frac{df}{dt} = Df^3 + E\lambda^2 f,\ D > 0$$

To get $\beta_\lambda < 0$, with $A > 0$, we need $f^2 \sim \lambda$. This means we can drop E term in β_f. With $D > 0$, Yukawa coupling f is not asymptotically free. Generalization to more than one fermion fields or more scalar fields conclusion remains the same.

3. Abelian gauge theory(QED)

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\frac{de}{dt} = \beta_e = \frac{e^3}{12\pi^2} + O(e^5)$$

$$scalar\ QED: \quad \frac{de}{dt} = \beta_e = \frac{e^3}{48\pi^2} + O(e^5)$$

Both are not asymptotically free.

4. Non-Abelian gauge theories are asymptotically free.

$$\mathcal{L} = -\frac{1}{2} T_r(F_{\mu\nu} F^{\mu\nu})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \qquad A_\mu = T_a A^a_\mu$$

with

$$[T_a, T_b] = if_{abc} T_c \ , \quad T_r(T_a, T_b) = \frac{1}{2}\delta_{ab}$$

The equation for effective coupling constant

$$\frac{dg}{dt} = \beta(g) = -\frac{g^3}{16\pi^2}(\frac{11}{3})t_2(V) < 0$$

where

$$t_2(V)\delta^{ab} = t_r[T_a(V) T_b(V)] \qquad t_2(V) = n \ \ for \ \ SU(n)$$

Thus this is *asymptotically free*. If in addition,there are fermions and scalars with representation, $T^a(F)$ and $T^a(s)$, then

$$\beta_g = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}t_2(V) + \frac{4}{3}t_2(F) + \frac{1}{3}t_2(s)\right]$$

where

$$t_2(F)\delta^{ab} = t_r(T^a(F)T^b(F))$$

$$t_2(S)\delta^{ab} = t_r(T^a(S)T^b(S))$$

Thus as long as $t_2(F)$,and $t_2(s)$ are not too large, we still have $\beta_g < 0$.

QCD

Quark model needs colors degrees of freedom to overcome paradoxes of simple quark model. On the other-hands, Bjroken scaling in deep inelastic scattering seems to require asymptotically free theory.

These suggest gauging the color degrees of freedom of quarks ⇒ Quantum chromodynamics

$$\mathcal{L}_{QCD} = -\frac{1}{2} t_r(G_{\mu\nu} G^{\mu\nu}) + \sum_k \bar{q}_k (i\gamma^\mu D_\mu - m_k) q_k$$

where

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ D_\mu q_k &= (\partial_\mu - igA_\mu) q_k \ , \qquad A_\mu = A_\mu^a \frac{\lambda^a}{2} \end{aligned}$$

$$\beta_g = \frac{-1}{16\pi^2}(11 - \frac{2}{3} n_f) = -bg^3$$

where n_f :number of flavors. The equation for effective coupling constant

$$\frac{d\bar{g}}{dt} = -b\bar{g}^3 \quad t = \ln \lambda$$

and the solution is

$$\bar{g}^2(t) = \frac{g^2}{1 + 2bg^2 t} \quad where \quad g = \bar{g}(g, 0)$$

For large momenta, $\lambda p_i, \bar{g}^2(t)$ decreases like $\ln \lambda$
Conveient to define

$$\alpha_s(Q^2) = \frac{\bar{g}^2(t)}{4\pi}$$

then we can write

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + 4\pi b \alpha_s(\mu^2) \ln(Q^2/\mu^2)}$$

Introduce Λ^2 by the relation,

$$\ln \Lambda^2 = \ln \mu^2 - \frac{1}{4\pi b \ \alpha_s(\mu^2)}$$

$$\textit{then} \quad \alpha_s(Q^2) = \frac{4\pi}{(11 - \frac{2}{3} n_f) \ln Q^2/\Lambda^2}$$

Thus the effective coupling constant $\alpha_s(Q^2)$ decreases slowly $\sim \frac{1}{\ln Q^2}$. QCD can make prediction about scaling violation (small) in the forms of integral over structure functions.

Quark confinements
Since $\alpha_s(Q^2)$ is small for large Q^2 , it is reasonable to believe that $\alpha_s(Q^2)$ is large for small Q^2 . If $\alpha_s(Q^2)$ is large enough between quarks so that quarks will never get out of the hadrons. This is called quark confinement. It is most attractive way to "explain" why quarks cannot be detected as free particles.

QCD and Flavor symmetry QCD Lagrangian is of the form

$$\mathcal{L}_{QCD} = -\frac{1}{2} t_r (G_{\mu\nu} G^{\mu\nu}) + \sum_k \bar{q}_k i\gamma^\mu D_\mu q_k + \sum_k \bar{q_k} m_k q_k$$

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ , \qquad A_\mu = A_\mu^a \frac{\lambda^a}{2} \\ D_\mu q_k &= (\partial_\mu - igA_\mu) q_k \ , \quad q_k = (u, d, s \cdots) \end{aligned}$$

Consider the simple case of 3 flavors.

$$q_k = (u, d, s)$$

$$\mathcal{L}_{QCD} = -\frac{1}{2} t_r (G_{\mu\nu} G^{\mu\nu}) + (\bar{u} i\gamma^\mu D_\mu u + \bar{d} i\gamma^\mu D_\mu d + \bar{s} i\gamma^\mu D_\mu s) + m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s$$

In the limit $m_u = m_d = m_s = 0, \$_{QCD}$, is invariant under $SU(3)_L \times SU(3)_R$ transformation

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}_L \to U_L \begin{pmatrix} u \\ d \\ s \end{pmatrix}_L \qquad \begin{pmatrix} u \\ d \\ s \end{pmatrix}_R \to U_R \begin{pmatrix} u \\ d \\ s \end{pmatrix}_R$$

However, hadron spectra shows only approximate $SU(3)$ symmetry, not $SU(3) \times SU(3)$ symmetry. We can reconcile this by the scheme $SU(3) \times SU(3)$ is broken spontaneously to $SU(3)$ so that particles group into $SU(3)$ multiplet. This would require 8 Goldstone bosons, which are massless. However in real world quark masses are not zero, these Goldstone bosons are not exactly massless. But if this symmetry breaking makes sense at all, these Goldstone bosons should be light. Thus we can identify them as pseudoscalar mesons. In other worlds, pseudoscalar mesons are "almost" Goldstone bosons.