Quantum Field Theory

Ling-Fong Li

National Center for Theoretical Science

1 Path Integral Quantization of Gauge Theories

Canonical quantization of gauge theory is difficult, because not all components of gauge fileds are real physical degree of freedom. To eliminate those components which are dependent, it is eaiser to use path integral quantization To see the difficulty, consider SU(2) Yang-Mills fields,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \qquad a = 1, 2, 3$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$$

The generating functional is

$$W[J] = \int [dA_\mu] \exp\{i \int d^4x [\mathcal{L} + \vec{J}_\mu \cdot \vec{A}^\mu]$$

The free-field part is

$$W_0[J] = \int [dA_\mu] \exp\{i \int d^4x [\mathcal{L}_0 + \vec{J}_\mu \cdot \vec{A}^\mu]\}$$

Write the free Lagrangian as,

$$\int d^4x \mathcal{L}_o(x) = -\frac{1}{4} \int d^4x (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial_\mu A^{a\nu} - \partial_\nu A^{a\mu})$$
$$= \frac{1}{2} \int d^4x A^a_\mu(x)(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) A^a_\nu(x)$$

The formula for the Gaussian integral is,

$$\int [d\phi] \exp[-\frac{1}{2} \langle \phi K \phi \rangle + \langle J \phi \rangle] \sim \frac{1}{\sqrt{\det K}} \exp \langle J K^{-1} J \rangle$$

However, in our case operator K

$$K_{\nu\mu}(x-y) = (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu)\delta^4(x-y)$$

is a projection operator, i.e.

$$\int d^4 y K_{\mu\nu}(x-y)\, K^\nu{}_\lambda(y-z) \propto K_{\nu\lambda}(x-z)$$

and has no inverse. Then Gaussian intergral diverges. $W_0(J)$ is singular is due to the gauge invariance which projects out the transverse gauge fields. In $W_0(J)$ we sum over all field configurations, including "orbits" that are related by gauge transformation. This over-counting is the root of the divergent integral. Thus we have to remove this "volume" of the orbit in the quantization.

Volume factor in gauge theory

Simple example

We shall use a 2-dimensional integral to illustrate the strategy to factor out the volume factor. Take a simple integral,

$$W = \int dxdye^{iS(x,y)} = \int d^2re^{iS(\vec{r})} \tag{1}$$

where $\vec{r} = (r, \theta)$. Suppose $S(\vec{r})$ is invariant under rotation,

$$S(\vec{r}) = S(\vec{r}_\phi), \qquad \text{with} \qquad \vec{r}_\phi = (r, \theta + \phi) \tag{2}$$

Then $S(\vec{r})$ is constant over (circular) orbit and W is proportional to the length of the orbit. So if we sum over contribution from inequivalent $S(\vec{r})'s$ we can simply divide out the volume factor corresponding to polar integration $\int d\theta = 2\pi$. We will use a more complicate prodedure which can be generalized to more general cases. Insert an identity,

$$1 = \int d\phi\delta\left(\theta - \phi\right)$$

into W given in Eq(1)

$$W = \int d\phi \int d^2re^{iS(\vec{r})}\delta\left(\theta - \phi\right) = \int d\phi W_\phi$$

Use the invarinat property $S(\vec{r}) = S(\vec{r}_\phi)$, we see that

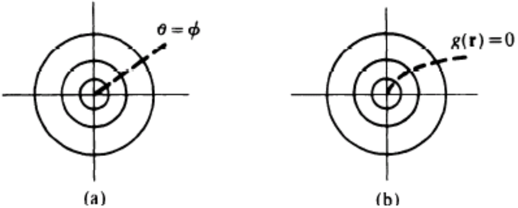
$$W_\phi = W_{\phi'}, \qquad \Longrightarrow \qquad W_\phi \text{ is independent of } \phi$$

and

$$W = \int d\phi W_\phi = W_\phi \int d\phi = 2\pi W_\phi$$

Here we can just drop volume factor 2π. Impose more complicate constraint,

$$g(\vec{r}) = 0 \tag{3}$$



which intersects each orbit only once. We need to compute $\left[\Delta_g\left(\vec{r}\right)\right]$ defined by

$$1 = \int d\phi \ \left[\Delta_g\left(\vec{r}\right)\right] \delta\left[g(\vec{r}_\phi)\right]$$

Write

$$\left[\Delta_g\left(\vec{r}\right)\right]^{-1} = \int d\phi \ \delta\left[g(\vec{r}_\phi)\right]$$

$\Delta_g\left(r\right)$ is rotational invariant,

$$\left[\Delta_g\left(\vec{r}_{\phi'}\right)\right]^{-1} = \int d\phi \ \delta\left[g(\vec{r}_{\phi+\phi'})\right] = \int d\phi" \ \delta\left[g(\vec{r}_{\phi"})\right] = \left[\Delta_g\left(\vec{r}\right)\right]^{-1}$$

Integrating over ϕ,

$$\Delta_g\left(\vec{r}\right) = \left.\frac{\partial g(\vec{r})}{\partial\theta}\right|_{g=0} \tag{4}$$

The integral is

$$W = \int d\phi W_\phi \qquad \text{with} \quad W_\phi = \int d^2 r e^{iS(\vec{r})}\delta\left[g(\vec{r}_\phi)\right]\Delta_g\left(\vec{r}\right) \tag{5}$$

Again, W_ϕ is rotational invariant and we can remove the voulume factor in Eq(5),

$$\begin{aligned} W_{\phi'} &= \int d^2 r e^{iS(\vec{r})}\delta\left[g(\vec{r}_{\phi'})\right]\Delta_g\left(\vec{r}\right) \\ &= \int d^2 r' e^{iS(\vec{r}')}\delta\left[g(\vec{r}_{\phi'})\right]\Delta_g\left(\vec{r}'\right) \qquad \text{with} \quad \vec{r}' = (r, \phi') \end{aligned}$$

Volume factor in Gauge Theories

The gauge theory is more complicate. But the principle is the same and it is useful to think of local gauge symmetry as the generalization of the rotational symmetry in the simple example .

Under the gauge transformation we have $\overrightarrow{A}_\mu \rightarrow \overrightarrow{A}_\mu^\theta$, where

$$\overrightarrow{A}_\mu \cdot \frac{\overrightarrow{\tau}}{2} \longrightarrow \overrightarrow{A}_\mu^\theta \cdot \frac{\overrightarrow{\tau}}{2} = U(\theta)[(\overrightarrow{A}_\mu \cdot \frac{\overrightarrow{\tau}}{2}) + \frac{1}{ig} U^{-1}(\theta)\partial_\mu U(\theta)]U^{-1}(\theta)$$

where

$$U(\theta) = \exp[\frac{-i\overrightarrow{\theta} \cdot \overrightarrow{\tau}}{2}]$$

This is analogous to rotational transformation given in Eq(2). We restrict the path integration to hypersurface which intersects each orbit once. If we choose the hypersurface as

$$f_a(\overrightarrow{A}_\mu) = 0, \quad a = 1, 2, 3 \tag{6}$$

so that the equation

$$f_a(\overrightarrow{A}_\mu^\theta) = 0$$

This is analogous to Eq (3). In the neighborhood of identity, we can write

$$U(\theta) = 1 + i\frac{\overrightarrow{\theta} \cdot \overrightarrow{\tau}}{2} + O(\theta^2)$$

The integration over group space can be chosen as

$$[d\theta] = \prod_{a=1}^{3} d\theta_a$$

Define

$$\Delta_f^{-1}[\overrightarrow{A}_\mu] = \int [d\theta(x)]\delta[f_a(\overrightarrow{A}_\mu^\theta)]$$

then

$$\Delta_f[A_f] = \det\, M_f \qquad \text{where} \quad (M_f)_{ab} = \frac{\delta f_a}{\delta \theta_b}$$

This is the generalization of the formula,

$$\int dx\,\delta\,(f\,(x)) = \left.\frac{1}{df\,/\,dx}\right|_{f=0}$$

Under infinitesimal gauge transformation

$$A_\mu^{\theta a} = A_\mu^a + \epsilon^{abc}\theta^b A_\mu^c - \frac{1}{g}\partial_\mu\theta^a$$

and the responce of the function f is

$$f_a(\overrightarrow{A}_\mu^\theta) = f_a(\overrightarrow{A}_\mu) + \int d^4y[M_f(x,y]_{ab}\theta_b(y) + O(\theta^2)$$

Again $\Delta_f[\overrightarrow{A}_\mu]$ is gauge invariant, as illustrated by the following simple calculation. From

$$\Delta_f^{-1}[\overrightarrow{A}_\mu] = \int [d\theta'(x)]\delta[f_a(\overrightarrow{A}_\mu^{\theta'})]$$

we get

$$\Delta_f^{-1}[\overrightarrow{A}_\mu^{\theta}] = \int [d\theta'(x)]\delta[f_a(\overrightarrow{A}_\mu^{\theta\theta'})] = \int [d(\theta(x)\theta'(x)]\delta[f_a(\overrightarrow{A}_\mu^{\theta\theta'})]$$
$$= \int [d\theta"(x)]\delta[f_a(\overrightarrow{A}_\mu^{\theta"})] = \Delta_f^{-1}[\overrightarrow{A}_\mu]$$

The path integral is

$$\int [d\overrightarrow{A}_\mu] \exp\{i\int \mathcal{L}(x)d^4x\} = \int [d\theta(x)] \left[d\overrightarrow{A}_\mu\right] \Delta_f(\overrightarrow{A}_\mu)\delta[f_a(\overrightarrow{A}_\mu^{\theta})] \exp\{i\int \mathcal{L}(x)d^4x\}$$
$$= \int [d\theta(x)] \left[d\overrightarrow{A}_\mu\right] \Delta_f(\overrightarrow{A}_\mu)\delta[f_a(\overrightarrow{A}_\mu)] \exp\{i\int \mathcal{L}(x)d^4x\}$$

We can now drop the "volume factor" $\int [d\theta(x)]$ to write the generating functional as

$$W_f[\vec{J}] = \int [d\overrightarrow{A}_\mu](\det M_f)\delta[f_a(\overrightarrow{A}_\mu)] \exp\{i\int d^4x[\mathcal{L}(x) + \overrightarrow{J}_\mu \cdot \overrightarrow{A}^\mu]\}$$

This is calles Faddeev-Popov ansatz and the factor det M_f is called the Faddeev-Popov determinant. This is the path integral suitable for quantization.

Faddeev-Popov Ghost

Write det M_f as

$$(\det M_f) \sim \int [dc][dc^+] exp\{i \int d^4x d^4y \sum c_a^+(x)[M_f(x,y)]_{ab} c_b(y)\}$$

where $c_a, c_b^\dagger$ are Grassman fields and are called Faddeev-Popov ghosts and they are not real physical degrees of freedoms. Then we can treat the Faddeev-Popov determinant as an additional term in the Lagrangian. We also want to convert $\delta[f_a(A_\mu)]$ into some effective Lagrangian form. Suppose we choose the gauge fixing term as ,

$$[f_a(\vec{A}_\mu)] = B_a(x)$$

$B_a(x)$ is some arbitrary function. Then the integral

$$\int [d\theta(x)] \Delta_f[\vec{A}_\mu] \delta[f_a(\vec{A}_\mu^\theta) - B_a(x)] = 1$$

will give the same $\Delta_f[A_\mu]$ as before. Note that

$$\int [dB_a(x)] exp\{-\frac{i}{2\xi} \vec{B}^2(x)\} \sim constant, \qquad \xi \text{ is arbitrary}$$

We can write

$$W[J] = \int [dA^a_\mu][dB_a(x)](\det M_f)\delta[f_a(\vec{A}_\mu) - B_a] \exp\{i \int d^4x[\mathcal{L}(x) - \vec{J}^\mu \cdot \vec{A}_\mu - \frac{1}{2\xi}\vec{B}^2(x)]\}$$

$$= \int [dA^a_\mu](\det M_f) \exp\{i \int d^4x[\mathcal{L}(x) - \vec{J}^\mu \cdot \vec{A}_\mu - \frac{1}{2\xi}[f^a(A_\mu)]^2]\},$$

Put all these together

$$W[J] = \int [dA^a_\mu][dc(x)][dc^\dagger(x)] \exp\{iS_{eff}[\vec{J}]\}$$

where the effective action is,

$$S_{eff}[\vec{J}] = S[\vec{J}] + S_{gf} + S_{FPG}$$

Here S_{gf} is the gauge fixing term,

$$S_{gf} = \frac{1}{2\xi} \int d^4x\{f_a[A_\mu(x)]\}^2$$

and S_{FPG} is the Faddev-Popov ghost term,

$$S_{FPG} = \int d^4x d^4y \sum_{a,b} c^\dagger_a(x)[M_f(x,y)]_{ab} c_b(y)$$

Covariant gauge

Themost common choice of the gauge fixing term is

$$f_a(A_\mu) = \partial^\mu A^a_\mu = 0$$

We now compute the Faddev-Popov determinan . Under infinitesimal gauge transformation,

$$U(\theta(x)) = 1 + \frac{i\vec{\theta}\cdot\vec{\tau}}{2} + O(\theta^2)$$

we get

$$A^{a\theta}_\mu = A^a_\mu + \epsilon^{abc}\theta^b(x)A^c_\mu(x) - \frac{1}{g}\partial_\mu\theta^a$$

Then

$$f^a(A^\theta_\mu) = f^a(A_\mu) + \partial^\mu[\epsilon^{abc}\theta^b(x)A^c_\mu(x) - \frac{1}{g}\partial_\mu\theta^a(x)] = f^a(A_\mu) + \int d^4y[M_f(x,y]_{ab}\theta^b(y)$$

with

$$[M_f(x,y)]_{ab} = -\frac{1}{g}\partial^\mu[\delta^{ab}\partial_\mu - g\epsilon^{abc}A^c_\mu]\delta^4(x-y)$$

Then

$$S_{gf} = -\frac{1}{2\xi}\int d^4x(\partial^\mu A_\mu)^2$$
$$S_{FPG} = \frac{1}{g}\int d^4x\sum_{a,b} c^+_a(x)\partial^\mu[\delta_{ab}\partial_\mu - g\epsilon_{abc}A^c_\mu]c_b(x)$$

We can use this to generate Feynman rule and do the calculation perturbatively if applicable.