Quantum Field Theory

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Grand Unification Theory

The standard model of $SU(3)_C \times SU(2)_L \times U(1)_Y$ are all based on gauge theory. Want to combine all these interactions as components of a single force; a theory with only one guage coupling.

SU(5) Model :1974 Georgi and Glashow

A general representation of $SU(5)$ in tensor notation transforms as,

$$\psi^{ij\cdots}_{kl\cdots} \longrightarrow U^i_m U^j_n U^s_k U^t_l \cdots \psi^{mn\cdots}_{st\cdots}$$

where

$$[U]^i_m = [\exp(i\alpha^a \lambda^a/2)]^i_m$$

is a 5×5 unitary matrix and $\{\lambda^a\}, a = 0, 1, 2, \cdots, 23$ is a set of 5×5 hermitian traceless matrices with ,

$$T(\lambda_a \lambda_b) = 2\delta_{ab}$$

For example,

$$\lambda^3 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \qquad \lambda^0 = \frac{1}{\sqrt{15}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

To obtain the $SU(3)_C \times SU(2)_L$ content of a representation, we identify first 3 of $SU(5)$ indices as color indices and the other 2 as $SU(2)_L$ indices,

$$i = (\alpha, r), \qquad \text{with} \quad \alpha = 1,2,3 \qquad r = 1,2 \tag{1}$$

Fermion content

In the standard model with one generation, the fermion content with respect to $SU(3)_C \times SU(2)_L$ are given by,

$$(\nu_e, e)_L \sim (\mathbf{1}, \mathbf{2}), \qquad e_L^+ \sim (\mathbf{1}, \mathbf{1}),$$

$$(u_\alpha, d_\alpha)_L \sim (\mathbf{3}, \mathbf{2}), \qquad u_L^{c\alpha} \sim (\mathbf{3}^*, \mathbf{1}), \qquad d_L^{c\alpha} \sim (\mathbf{3}^*, \mathbf{1})$$

where we have used the relaitons

$$\psi^c = C\gamma^0\psi^*, \qquad (\psi_R)^c = (\psi^c)_L \equiv \psi_L^c$$

The $SU(3)_C \times SU(2)_L$ contents of simple $SU(5)$ representations are ;

$$\begin{array}{ccc} \text{lowest rep} & \psi_i & \mathbf{5} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) \\ \text{lowest conjugate} & \psi^i & \mathbf{5}^* = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) \\ \text{anti-symm} & \psi_{ij} & \mathbf{10} = (\mathbf{3}^*, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}) \end{array}$$

Comparing with first generation fermions, we see that they can be accomodated as $\mathbf{5}^* + \mathbf{10}$ representation of $SU(5)$,

$$\mathbf{5}^* : \left(\psi^i\right)_L = \left(d^{c1}, d^{c2}, d^{c3}, e^-, \nu\right)_L$$

and

$$10:(\chi_{ij})_L=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & u^{c3} & -u^{c2} & u_1 & d_1 \\ -u^{c3} & 0 & u^{c1} & u_2 & d_2 \\ u^{c2} & -u^{c1} & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & -0 \end{pmatrix}$$

Charge Quantization

One consequence of the $SU(5)$ unification is a explanation for charge quantization. In general, if unification group is simple, charge quantization follows, because eigenvalues of a non-Abelian group are discrete while Abelian $U(1)$ group are continuous.In $SU(3)_C\times SU(2)_L$ model electric charge operator can be written as

$$Q=T_3+\frac{Y}{2}$$

express this relation in terms of generators of $SU(5)$. Write

$$Q=T_3+cT_0$$

It is straightforward to see that

$$c=-\sqrt{\frac{5}{3}}$$

Gauge bosons

The $SU(5)$ adjoint representation A_i^j has dimension 24 and the $SU(3)_C \times SU(2)_L$ content is

$$\mathbf{24} = (\mathbf{8},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{2}) + (\mathbf{3}^*,\mathbf{2})$$

A_β^α	$(\mathbf{8},\mathbf{1})$	$SU(3)_C$ color gluons
A_s^r	$(\mathbf{1},\mathbf{3})$	$SU(2)_L$ weak bosons
$A_\alpha^\alpha - A_r^r$	$(\mathbf{1},\mathbf{1})$	$U(1)$ weak boson
A_α^r	$(\mathbf{3},\mathbf{2})$	Lepto-quark
A_r^α	$(\mathbf{3}^*,\mathbf{2})$	Lepto-quark

The lepto-quark

$$A_\alpha^r = (X_\alpha, Y_\alpha), \qquad A_r^\alpha = \begin{pmatrix} X^\alpha \\ Y^\alpha \end{pmatrix}$$

have fractional charges

$$Q(X) = -\frac{4}{3}, \qquad Q(Y) = -\frac{1}{3}$$

and are important in the Baryon number violation. Put all $SU(5)$ gauge bosons in 5×5 matrix

$$A_\mu = \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2}$$

we get

$$A = \begin{pmatrix} G_1^1 & G_2^1 & G_3^1 & X_1 & Y_1 \\ G_1^2 & G_2^2 & G_3^2 & X_2 & Y_2 \\ G_1^3 & G_2^3 & G_3^3 & X_3 & Y_3 \\ X^1 & X^2 & X^3 & W^3 + B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -W^3 + B \end{pmatrix} \tag{2}$$

Spontaneous symmetry breaking

Spontaneous symmetry breaking in two stages, characterized by v_1 and v_2

$$SU(5) \xrightarrow{v_1} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow{v_2} SU(3)_C \times U(1)_{EM}$$

where $v_1 \gg v_2$. In 1st stage, X, Y masses being superheavy, $M_{X,Y} \gg M_{W,Z}$. This can be achieved with scalars in adjoint $\left(H_j^i\right)$ for v_1 and vector (ϕ_i)representations for v_2. The general $SU(5)$ −invariant 4-th order potential is,

$$V(H,\phi) = V_1(H) + V_2(\phi) + \lambda_4 tr\left(H^2\right)\left(\phi^\dagger \phi\right) + \lambda_5 \left(\phi^\dagger H^2 \phi\right)$$

with

$$V_1(H) = -m_1^2 tr\left(H^2\right) + \lambda_1 \left[tr\left(H^2\right)\right]^2 + \lambda_2 tr\left(H^4\right)$$

$$V_2(\phi) = -m_2^2\left(\phi^\dagger \phi\right) + \lambda_3 \left(\phi^\dagger \phi\right)^2$$

H is 5×5 traceless hermitian matrix. Use discrete symmetry $H \to -H$ and $\phi \to -\phi$ to remove cubic terms. We first minimize the potential $V_1(H)$. It turns out that for $\lambda_2 > 0$ and $\lambda_1 > -\frac{7}{30}$, $V_1(H)$ has an extremum at $H = \langle H \rangle$ with

$$\langle H \rangle = v_1 \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

where

$$v_1^2 = \frac{m_1^2}{[60\lambda_1 + 14\lambda_2]}$$

and symmetry breaking is

$$SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$$

and gauge bosons X, and Y obtain masses $\propto v_1$. As we will see later v_1 could be of order of 10^{15} Gev or so.

The fact H develops VEV also effects the ϕ system. The color triplet $\phi_t : (\mathbf{3}, \mathbf{1})$ and flavor doublet $\phi_d : (\mathbf{1}, \mathbf{2})$ components of $\phi = (\phi_t, \phi_d)$ acquires mass terms,

$$m_t^2 = -m_2^2 + (30\lambda_4 + 4\lambda_5)\, v_1^2 \qquad (3)$$

$$m_d^2 = -m_2^2 + (30\lambda_4 + 9\lambda_5)\, v_1^2 \qquad (4)$$

after first stage of symmetry breaking all masses are of order of v_1 , superheavy. For the second stage of symmetry breaking we need a $SU(2)$ doublet to break the symmetry at energy of order of 250 Gev. Here we assume that "somehow" the m_d^2 in Eq (4) is much smaller than v_1^2 and will survive to low energy(~ 250 Gev) as the superheavy particle (with masses of order v_1) decouple. The relevant physics is described by the effective potential,

$$V_{eff}(\phi_d) = -m_d^2 \left(\phi_d^\dagger \phi_d\right) + \lambda_3 \left(\phi_d^\dagger \phi_d\right)^2$$

which produces symmetry breaking

$$SU(2) \times U(1) \longrightarrow U(1)$$

$$\phi_d = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ v_2 \end{array} \right), \qquad v_2 = \sqrt{\frac{m_d^2}{\lambda_3}} \sim 250 \ Gev$$

This feature where $v_1 \gg v_2$ is usually called the **gauge hierachy**.

Coupling constant unification

In standard model, elecromagnetic, for energies $\leq 10^2 Gev$ there are 3 different coupling constants: g_s, g, and g' for , $SU(3)_C, SU(2)_L$ and $U(1)_Y$ respectively. The grand unified theory unifies these into one coupling constant. In spontaneous symmetry breaking; X, Y gauge bosons of $SU(5)$ acquire masses and decouple from the coupling constant renormalization.This leads to 3 different coupling constants. Since the energy dependence of coupling constants is only logarithmic, unification scale M_X is expected to be many orders of magnitude larger than $10^2 Gev$.

The covariant derivative for $SU(5)$ is

$$D_\mu = \partial_\mu + ig_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2}$$

and for $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$D_\mu = \partial_\mu + ig_s \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} + ig \sum_{r=1}^{3} W_\mu^r \frac{\lambda^r}{2} + ig' B_\mu \frac{Y}{2}$$

All non-Abelian groups here are normalized as $Tr(\lambda^a \lambda^b) = 2\delta^{ab}$ and we have

$$g_5 = g_3 = g_2 = g_1 \tag{5}$$

with

$$g_3 = g_s, \qquad g_2 = g$$

The coupling g_1 is that of the Abelian $U(1)$ subgroup. Thus

$$ig_1\lambda^0 A_\mu^0 = ig' Y B_\mu$$

and A_μ^0 is identified with B_μ gauge field. Note that

$$Y = \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

From this we get

$$Y = -\sqrt{\frac{5}{3}}\lambda^0, \qquad g' = -\sqrt{\frac{3}{5}}g_1$$

The weak mixing angle is then

$$\sin^2\theta_W = \frac{g'^2}{g^2 + g'^2} = \frac{3}{8} \tag{6}$$

The relations given Eqs (5,6) are valid at unification scale. To compare them with experimental data we need to evolve them down to low energy $\sim 10^2 Gev$. The evolution of the $SU(n)$ coupling constant is of the form,

$$\frac{dg_n}{d(\ln\mu)} = -b_n g_n^3$$

where

$$b_n = \frac{1}{48\pi^2}\left(11n - 2N_F\right) \qquad \text{for} \quad n \geq 2$$

$$b_1 = -\frac{N_F}{24\pi^2}$$

Then we get

$$b_n - b_1 = \frac{11n}{48\pi^2}$$

In our case, the solution for the effective coupling constants are

$$\frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2(\mu_0)} + 2b_1 \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{g_2^2(\mu)} = \frac{1}{g_2^2(\mu_0)} + 2b_2 \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{g_3^2(\mu)} = \frac{1}{g_3^2(\mu_0)} + 2b_3 \ln\left(\frac{\mu}{\mu_0}\right)$$

In terms of more familiar parameters,

$$\frac{g_1^2(\mu)}{4\pi} = \left(\frac{5}{3}\right)\frac{\alpha(\mu)}{\cos^2\theta_W}, \qquad \frac{g_2^2(\mu)}{4\pi} = \frac{\alpha(\mu)}{\sin^2\theta_W}, \qquad \frac{g_3^2(\mu)}{4\pi} = \alpha_s(\mu)$$

we get

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_5} + 8\pi b_3 \ln\left(\frac{\mu}{M_X}\right)$$

$$\frac{\sin^2\theta_W}{\alpha(\mu)} = \frac{1}{\alpha_5} + 8\pi b_2 \ln\left(\frac{\mu}{M_X}\right)$$

$$\frac{3}{5}\frac{\cos^2\theta_W}{\alpha(\mu)} = \frac{1}{\alpha_5} + 8\pi b_1 \ln\left(\frac{\mu}{M_X}\right)$$

where we have used

$$g_1(M_X) = g_2(M_X) = g_2(M_X) = g_5, \qquad \text{and} \quad \frac{g_5^2}{4\pi} = \alpha_5$$

Taking a linear combination to eliminate $\ln\left(\frac{\mu}{M_X}\right)$ to get

$$\frac{2}{\alpha_s} - \frac{3}{\alpha}\sin^2\theta_W + \frac{3}{5\alpha}\cos^2\theta_W = 8\pi\left[2(b_3 - b_1) - 3(b_2 - b_1)\right]\ln\left(\frac{\mu}{M_X}\right) = 0$$

This implies

$$\sin^2\theta_W = \frac{1}{6} + \frac{5\alpha(\mu)}{9\alpha_s(\mu)}$$

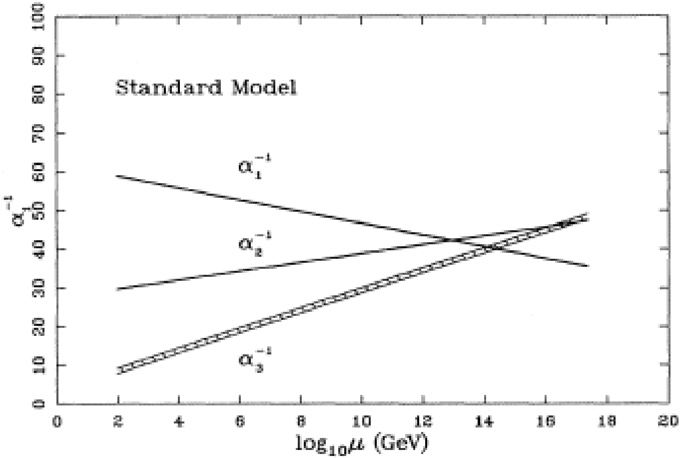
Using the measured values of $\alpha(\mu)$ and $\alpha_s(\mu)$ we get

$$\sin^2\theta_W \simeq .21$$

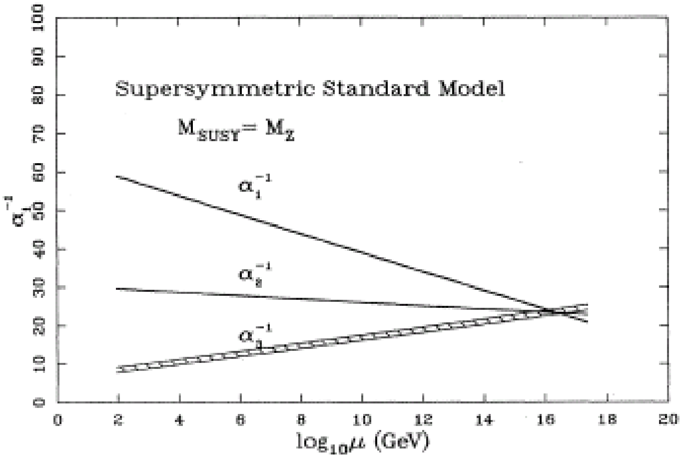
and

$$M_X \simeq 4\times 10^{14}\,Gev$$

Or using the measured values of $\sin^2\theta_W$, we get



This shows that these 3 coupling constants do not unify very well. It turns out that if we use supersymmetric version of unification we get



This has become one of the motivation for supersymmetry.

Baryon number violation

The gauge couplings of $\mathbf{5}^*+\mathbf{10}$ fermions (ψ^i, χ_{ij}), are of the form, Using the gauge boson matrix in Eq (2)

$$\begin{aligned} g\bar{\psi}\gamma^\mu A_\mu^T \psi + trg\bar{\chi}\gamma^\mu\{A_\mu, \chi\} &= -\sqrt{\frac{1}{2}} g W_\mu^\dagger \left(\bar{\nu}\gamma^\mu e + \bar{u}_\alpha \gamma^\mu e_\alpha\right) \\ &+\sqrt{\frac{1}{2}} g X_{\mu\alpha}^a [\varepsilon^{\alpha\beta\gamma}\bar{u}_\alpha\gamma^\mu q_{\beta a} + \varepsilon^{ab}(\bar{q}_{\alpha b}\gamma^\mu e^+ - \bar{l}_b\gamma^\mu d_\alpha^c)] \end{aligned}$$

Note that X bosons couple to two-fermion channels with different baryon numbers. In one case, they couple to quarks and leptons ($B = \frac{1}{3}$); in other case they transform quarks to antiquark ($B = \frac{2}{3}$).

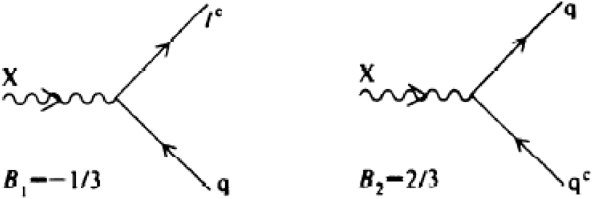


FIG. 14.4. X bosons as leptoquarks and diquarks.

Consequently , mediation of $X - boson$, a $B = -\frac{1}{3}$ channel can be converted into a $B = \frac{2}{3}$ one, a baryon number violation process. Since M_X is very heavy, effective 4-fermion local interaction for baryon violating processes is,

$$\mathcal{L}_{\Delta B=1} = \frac{g^2}{2M_X^2}\varepsilon^{\alpha\beta\gamma}\varepsilon^{ab}(\bar{u}^c_\alpha\gamma^\mu q_{\beta a})(\bar{d}^c_\alpha\gamma_\mu l_b + \bar{e}^+\gamma_\mu q_{\alpha b})$$

This will give following decays,

$$p \longrightarrow e^+\pi^0, \qquad e^+\omega, \qquad \cdots$$

To calculate these decay rates needs to renormalize these effective Lagrangian from M_X down to 1 Gev and use some hadronic model to compute the hadronic matrix elements. Note the factor M_X^2 in the denominator which makes the decay rates very small because $M_X \sim 10^{15}Gev$ or more. In the 80's the proton decay experiments have been actively pursued and none has been found. The best limit is

$$\tau(p \longrightarrow e^+\pi^0) \geq 1.6 \times 10^{33} \text{ years}$$

Baryon number asymmetry in the universe

Universe seems to be made out of mostly matter and no anti-matter. In hot Big Bang Model, the matter and anti-matter are produced in equal amount. How this matter- antimatter symmetric situation can evolve into matter- antimatter asymmetric universe? One quantitative measure is ratio of baryon number density n_B to the Cosmic Background Radiation (CMB) photon density n_γ,

$$\eta = \frac{n_B}{n_\gamma} \simeq (6.1 \pm 0.2) \times 10^{-10}$$

How this matter- antimatter symmetric situation can evolve into matter- antimatter asymmetric universe?

Sakharov has came up with 3 conditions needed to generate this asymmetry,

1. Baryon number violation
   If baryon number were conserved, then initial $n_B = 0$ of hot Big Bang model can not change as the universe evolves.

2. *C and CP violations*

   For a baryon violating reaction, $X \longrightarrow qq$, there will also be a mirror processes $\bar{X} \longrightarrow \bar{q}\bar{q}$ creating exactly negative amount of n_B, no net baryon number if these two processes can occur with same rate. Thus C and CP violations are needed to ge i.e.,
   $\Gamma(X \longrightarrow qq) \neq \Gamma\left(\bar{X} \longrightarrow \bar{q}\bar{q}\right)$.

3. Out of thermal equilibrium

   CPT invariance requires particle and anti-particle to have the same mass, hence to be equally weighted in the Boltzmann distribution; thus no CPT invariant interactions can generate a non-zero baryon number density.

GUTs, such as $SU(5)$, togather with expansion of the universe can satisfy all these conditions. Unfortunately the CP violation in the Standard Model is not large enough to account for observed aymmetry $\eta \sim 10^{10}$.