Quantum Gravitational Contributions to Gauge Theories

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Outline

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Introduction

- Einstein gravity theory is not renormalizable from the perspective of quantum field theory.
- Quantum effects of gravity become significant only at Planck scale, and it may be possible to treat it as an effective field theory at low energy scale.
- Robinson and Wilczek claimed that quantum gravity can correct gauge couplings with power-law running and render all gauge theories asymptotically free. Later, further analysis by other authors suggested that the previous result was gauge dependent.

The Lagrangian

We begin with the action of Einstein-Yang-Mills theory

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{\kappa^2} R - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} \mathcal{F}^a_{\mu\nu} \mathcal{F}^a_{\alpha\beta} \right] \quad (1)$$

where R is Ricci scalar and $\mathcal{F}^a_{\mu\nu}$ is the Yang-Mills fields strength $\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu]$. With the minus-dimension coupling constant $\kappa = \sqrt{16\pi G}$. Usually, one expands the metric tensor around a background metric $\bar{g}_{\mu\nu}$ and treats graviton field as quantum fluctuation $h_{\mu\nu}$ propagating on the background space-time, $\bar{g}_{\mu\nu}$.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \quad g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu_\alpha h^{\alpha\nu} + \ldots \quad (2)$$

$$\sqrt{-g} = \sqrt{-\bar{g}}[1 + \frac{1}{2}\kappa h - \frac{1}{4}\kappa^2 (h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^2) \ldots] \quad (3)$$

Gauge-fixing condition

Let us set $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. $h_{\mu\nu}$ is interpreted as graviton field, fluctuating in flat space-time. The lagrangian can be arranged to different orders of $h_{\mu\nu}$ or κ. In the de Donder harmonic gauge

$$\chi^\mu = \partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h^\nu_\nu = 0$$

Graviton propagator has a simple form

$$P_G^{\mu\nu\rho\sigma}(k) = \frac{i}{k^2}[g^{\nu\rho}g^{\mu\sigma} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma}] \tag{4}$$

For simplicity, in the following, the metric $g^{\mu\nu}$ is understood as $\eta^{\mu\nu}$.

Faddeev-Popov factor

Gauge fixing condition generally is accompanied with ghost. For instance, gauge condition

$$\chi^{\alpha}[g_{\mu\nu}] = 0$$

introduces ghost with its action as

$$\mathcal{L}_{\mathrm{gh}} = \bar{c}_{\alpha} Q^{\alpha}{}_{\beta} c^{\beta}, \tag{5}$$

where $Q^{\alpha}{}_{\beta}$ is the Faddeev-Popov factor,

$$Q^{\alpha}{}_{\beta} = \frac{\delta\chi^{\alpha}}{\delta\epsilon^{\beta}}.$$

With harmonic gauge, the ghosts do not interact with other field, we can ignore them.

β function and Counterterms

Using the traditional Feynman diagram calculations, we can get the β function by evaluating two and three point functions of gauge fields. These Green functions are generally divergent, so counter-terms are needed to cancel these divergences. The relevant counter-terms to the β function are

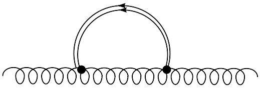
$$\begin{aligned}
&T^{\mu\nu} = i\delta_{ab}Q^{\mu\nu}\delta_2, \quad T^{\mu\nu\rho} = gf^{abc}V^{\mu\nu\rho}_{qkp}\delta_1 \\
&Q^{\mu\nu} \equiv q^\mu q^\nu - q^2 g^{\mu\nu} \\
&V^{\mu\nu\rho}_{qkp} \equiv g^{\nu\rho}(q-k)^\mu + g^{\rho\mu}(k-p)^\nu + g^{\mu\nu}(p-q)^\rho
\end{aligned} \tag{6}$$

The β function is given by

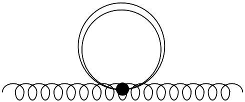
$$\beta(g) = g\mu\frac{\partial}{\partial\mu}(\frac{3}{2}\delta_2 - \delta_1) \tag{7}$$

Feynman Diagrams

At one loop level, diagrams are listed below. Also, we only have to keep the quadratic divergences, because logarithmic ones will only contribute to high order operator and then not to β function.



(a)



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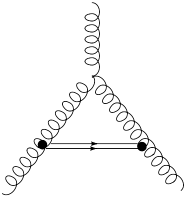

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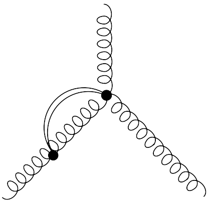
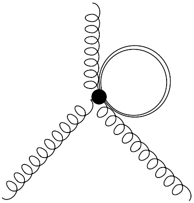


(d)



(e)

Two-point

The two point functions from Fig. 1(a) and Fig. 1(b) are found in terms of ILIs to be

$$
\begin{aligned}
\mathrm{T}^{(a)\mu\nu} &= 2\kappa^2 \int \mathrm{dx} \Big[\mathrm{Q}^{\mu\nu} \left[\mathcal{I}_2 + \mathrm{q}^2 (3\mathrm{x}^2 - \mathrm{x}) \mathcal{I}_0 \right] \\
&+ \mathrm{q}^\mu \mathrm{q}_\rho \mathcal{I}_2^{\nu\rho} + \mathrm{q}^\nu \mathrm{q}_\rho \mathcal{I}_2^{\mu\rho} - \mathrm{g}^{\mu\nu} \mathrm{q}_\rho \mathrm{q}_\sigma \mathcal{I}_2^{\rho\sigma} - \mathrm{q}^2 \mathcal{I}_2^{\mu\nu} \Big] (\mathcal{M}_\mathrm{q}^2) \\
\mathrm{T}^{(b)\mu\nu} &= -3\kappa^2 \mathrm{Q}^{\mu\nu} \mathcal{I}_2(0)
\end{aligned}
\tag{8}
$$

where we have defined

$$
\begin{aligned}
\mathcal{I}_2(\mathcal{M}^2) &\equiv \int \mathrm{d}^4\mathrm{k} \frac{1}{\mathrm{k}^2 - \mathcal{M}^2}, \\
\mathcal{I}_{2\mu\nu}(\mathcal{M}^2) &\equiv \int \mathrm{d}^4\mathrm{k} \frac{\mathrm{k}_\mu \mathrm{k}_\nu}{(\mathrm{k}^2 - \mathcal{M}^2)^2}
\end{aligned}
\tag{9}
$$

Three-piont

Three point functions from Fig.1(d) and 1(e) are found, when keeping only the quadratically divergent terms, to be

$$
\begin{aligned}
\mathrm{T}^{(\mathrm{d})\mu\nu\rho} &= \mathrm{i}\mathrm{g}\kappa^2\Bigg\{ -\mathrm{V}^{\mu\nu\rho}_{\mathrm{qkp}}\mathcal{I}_2(0) + \\
&\int \mathrm{dx}\Bigg[\Bigg(\mathrm{g}^{\mu\nu}\mathrm{q}_\sigma\mathcal{I}_2^{\rho\sigma} - \mathrm{g}^{\nu\rho}\mathrm{q}_\sigma\mathcal{I}_2^{\mu\sigma} + \mathrm{q}^\rho\mathcal{I}_2^{\mu\nu} - \mathrm{q}^\mu\mathcal{I}_2^{\nu\rho}\Bigg)(\mathcal{M}_\mathrm{q}^2) \\
&+\Bigg(\mathrm{g}^{\nu\rho}\mathrm{k}_\sigma\mathcal{I}_2^{\mu\sigma} - \mathrm{g}^{\rho\mu}\mathrm{k}_\sigma\mathcal{I}_2^{\nu\sigma} + \mathrm{k}^\mu\mathcal{I}_2^{\nu\rho} - \mathrm{k}^\nu\mathcal{I}_2^{\rho\mu}\Bigg)(\mathcal{M}_\mathrm{k}^2) \\
&+\Bigg(\mathrm{g}^{\rho\mu}\mathrm{p}_\sigma\mathcal{I}_2^{\nu\sigma} - \mathrm{g}^{\mu\nu}\mathrm{p}_\sigma\mathcal{I}_2^{\rho\sigma} + \mathrm{p}^\nu\mathcal{I}_2^{\rho\mu} - \mathrm{p}^\rho\mathcal{I}_2^{\mu\nu}\Bigg)(\mathcal{M}_\mathrm{p}^2)\Bigg]\Bigg\} \\
\mathrm{T}^{(\mathrm{e})\mu\nu\rho} &= 3\mathrm{i}\mathrm{g}\kappa^2\mathrm{V}^{\mu\nu\rho}_{\mathrm{qkp}}\mathcal{I}_2(0)
\end{aligned}
\tag{10}
$$

with $\mathcal{M}_\mathrm{q}^2 = \mathrm{x}(\mathrm{x}-1)\mathrm{q}^2$. Contraction is performed by using FeynCalc.

Regularization schemes

To handle with the divergent loop momentum integral, we need a regularization scheme. The lesson from quantum field theory tells us two guide rules.

Regularization schemes	Gauge invariance	Divergence behaviour
Cut-off R	$\times$	$\surd$
Dimensional R	$\surd$	$\times$
Loop R	$\surd$	$\surd$

Cut-off Regularization

Now we shall apply the different regularization schemes to the divergent ILIs. In cut-off regularization, when keeping only quadratically divergent terms, one has

$$\mathcal{I}_2^{R\mu\nu} = \frac{1}{4} g^{\mu\nu} \mathcal{I}_2^R, \quad \mathcal{I}_2^R \simeq \frac{i}{16\pi^2} g^{\mu\nu} \Lambda^2 \tag{11}$$

The resulting two and three point functions are

$$\begin{aligned} T_{\text{cutoff}}^{(a+b)\mu\nu} &\equiv T_{\text{cutoff}}^{(a)\mu\nu} + T_{\text{cutoff}}^{(b)\mu\nu} \\ &\approx 2Q^{\mu\nu}\kappa^2 \int dx \left[\frac{1}{2} \frac{i}{16\pi^2} \Lambda^2 + \left[\frac{i}{16\pi^2} \Lambda^2 - \frac{3}{2} \frac{i}{16\pi^2} \Lambda^2 \right] \right] = 0 \\ T_{\text{cutoff}}^{(d+e)\mu\nu\rho} &\equiv T_{\text{cutoff}}^{(d)\mu\nu\rho} + T_{\text{cutoff}}^{(e)\mu\nu\rho} \approx 0 \end{aligned}$$

which agrees with the result obtained by Ebert et.al.

Dimensional Regularization

In dimensional regularization, where $\mathcal{I}_2^R(0) = 0$ and $\mathcal{I}_{2\mu\nu}^R = \frac{1}{2} g_{\mu\nu} \mathcal{I}_2^R$, the two and three point functions are found to be

$$T_{DR}^{(a+b)\mu\nu} \approx 4\kappa^2 Q^{\mu\nu} \int dx \mathcal{I}_2^R(\mathcal{M}_q^2), \tag{12}$$

$$T_{DR}^{(d+e)\mu\nu\rho} = 2ig\kappa^2 \int dx \Big[(g^{\mu\nu} q^\rho - q^\mu g^{\nu\rho}) \mathcal{I}_2^R(\mathcal{M}_q^2) \tag{13}$$

$$+ (g^{\nu\rho} k^\mu - k^\nu g^{\rho\mu}) \mathcal{I}_2^R(\mathcal{M}_k^2) + (g^{\rho\mu} p^\nu - p^\rho g^{\mu\nu}) \mathcal{I}_2^R(\mathcal{M}_p^2) \Big]$$

where the regularized quadratic divergence in dimensional regularization behaves as the logarithmic one

$$\mathcal{I}_2^R(\mathcal{M}_q^2)|_{DR} = -\frac{-i}{16\pi^2} \mathcal{M}_q^2 [\frac{2}{\varepsilon} - \gamma_E + 1 + O(\varepsilon)] \tag{14}$$

Loop Regularization

We now make a calculation by using loop regularization. With the consistency condition $\mathcal{I}^R_{2\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{I}^R_2$, we obtain for two and three point functions

$$\begin{aligned} T^{(a+b)\mu\nu}_{LR} &= 2\kappa^2 Q^{\mu\nu} \int dx \Big[-\frac{3}{2}\mathcal{I}^R_2(0) + 2\mathcal{I}^R_2(\mathcal{M}^2_q) \\ &+ q^2(3x^2 - x)\mathcal{I}^R_0(\mathcal{M}^2_q) \Big] \end{aligned} \tag{15}$$

$$\begin{aligned} T^{(d+e)\mu\nu\rho}_{LR} &= 2ig\kappa^2 \int dx \Big[\frac{1}{2} V^{\mu\nu\rho}_{qkp} \mathcal{I}^R_2(0) + (g^{\mu\nu}q^\rho - q^\mu g^{\nu\rho})\mathcal{I}^R_2(\mathcal{M}^2_q) \\ &+ (g^{\nu\rho}k^\mu - k^\nu g^{\rho\mu})\mathcal{I}^R_2(\mathcal{M}^2_k) + (g^{\rho\mu}p^\nu - p^\rho g^{\mu\nu})\mathcal{I}^R_2(\mathcal{M}^2_p) \Big] \end{aligned} \tag{16}$$

β function

from which we can directly read off the two-point and three-point counter-terms δ_2^κ and δ_1^κ respectively(only keep the leading quadratically divergent part):

$$\delta_2^\kappa = \kappa^2 \frac{1}{16\pi^2}\left[\mathrm{M}_c^2 - \mu^2\right], \; \delta_1^\kappa = \kappa^2 \frac{1}{16\pi^2}\left[\mathrm{M}_c^2 - \mu^2\right]$$

we obtain the gravitational corrections to the gauge β function

$$\Delta\beta^\kappa = -\mathrm{g}\kappa^2 \frac{\mu^2}{16\pi^2} \tag{17}$$

Running of couplings

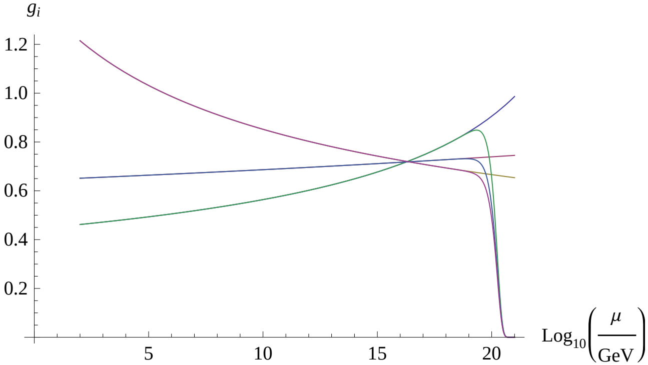


Figure: An illustration of gravitational contributions to the running of gauge couplings in the MSSM

Background Field Method

The method is used to calculate the effective action, Γ

$$\exp\frac{i}{\hbar}\Gamma[\bar{\varphi}] = \int D\mu[\varphi]\exp\frac{i}{\hbar}\left[S[\varphi] - \frac{\delta\Gamma}{\delta\bar{\varphi}^i}(\varphi^i - \bar{\varphi}^i)\right] \quad (18)$$

$$\begin{aligned}\Gamma[\bar{\varphi}] &= S[\bar{\varphi}] - \ln\det Q_{\alpha\beta}[\bar{\varphi}] \\ &\quad + \frac{1}{2}\ln\det\left(S[\bar{\varphi}]_{,ij} + \frac{1}{2\Omega}K^i_\alpha[\bar{\varphi}]K^\alpha_j[\bar{\varphi}]\right)\end{aligned} \quad (19)$$

In Einstein-Gauge system,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + a_\mu \quad (20)$$

coupling renormalization constant Z_g is connected to the gauge field renormalization constant $Z_A = 1 + \delta_A$ with $Z_g Z_A^{1/2} = 1$ in Background-Field Gauge, then β function is

$$\beta^\kappa_g = \mu\frac{\partial}{\partial\mu}g = \mu\frac{\partial}{\partial\mu}Z_g^{-1}g^0 = \frac{1}{2}g\mu\frac{\partial}{\partial\mu}\delta_A \quad (21)$$

Vilkovisky-DeWitt Formalism

The diagrammatic approach and traditional background field method share a gauge condition dependent problem. To overcome the gauge condition problem, we shall us Vilkovisky-DeWitt Formalism. The effective action is modified to

$$\exp\frac{\mathrm{i}}{\hbar}\hat{\Gamma}[\nu^{\mathrm{i}};\varphi_*]=\int \mathrm{D}\mu[\varphi]\exp\frac{\mathrm{i}}{\hbar}\left[\mathrm{S}[\varphi]-\frac{\delta\hat{\Gamma}}{\delta\nu^{\mathrm{i}}}(\sigma^{\mathrm{i}}(\varphi_*;\varphi)-\nu^{\mathrm{i}})\right] \tag{22}$$

where $\nu^{\mathrm{i}}\equiv\langle\sigma^{\mathrm{i}}(\varphi_*;\varphi)\rangle$ and the world function,

$$\sigma[\varphi_\star;\varphi]=\frac{1}{2}(\text{length of geodesic from } \varphi_\star \text{to } \varphi)^2.$$

The metric, $G_{ij}[\varphi]$

let $G_{ij}[\varphi]$ denote the metric of the field space, then at one-loop order, the effective action is given by

$$\begin{aligned}\hat{\Gamma}[\bar{\varphi}] &= S[\bar{\varphi}] - \ln\det Q_{\alpha\beta}[\bar{\varphi}] \\ &+ \frac{1}{2}\ln\det\left(\nabla^i\nabla_j S[\bar{\varphi}] + \frac{1}{2\Omega}\chi_{\alpha}^{,i}[\bar{\varphi}]\chi^{\alpha}_{,j}[\bar{\varphi}]\right)\end{aligned} \tag{23}$$

with $\nabla_i\nabla_j S[\bar{\varphi}] = S_{,ij}[\bar{\varphi}] - \bar{\Gamma}^k_{ij}S_{,k}[\bar{\varphi}]$. Here the connection $\bar{\Gamma}^k_{ij}$ is determined by $G_{ij}[\varphi]$. The metric is chosen to be

$$G_{g_{\mu\nu}(x)g_{\rho\sigma}(x')} = \frac{1}{\kappa^2}|g(x)|^{\frac{1}{2}}\left(g^{\mu(\rho}g^{\sigma)\nu} - \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\right)\delta(x,x') \tag{24}$$

$$G_{A_\mu(x)A_\nu(x')} = |g(x)|^{\frac{1}{2}}g^{\mu\nu}(x)\delta(x,x') \tag{25}$$

Landau-DeWitt gauge

The Vilkovisky-DeWitt formalism is applicable for any gauge condition. In Landau-DeWitt gauge, the calculation is much simpler. Landau-DeWitt gauge conditions $(\omega = 1)$ is determined by the gauge transformation and in $\mathrm{U}(1)$ gauge it reads

$$\begin{aligned} \chi_\lambda &= \frac{2}{\kappa}(\partial^\mu h_{\mu\lambda} - \frac{1}{2}\partial_\lambda h) + \omega(\bar{A}_\lambda \partial^\mu a_\mu + a^\mu \bar{F}_{\mu\lambda}), \\ \chi &= -\partial^\mu a_\mu. \end{aligned} \tag{26}$$

And in this case we can replace $\bar{\Gamma}^k_{ij}$ with Γ^k_{ij}, where Γ^k_{ij} is the Christoffel connection determined by

$$\nabla^i G_{jk} = 0$$

Parameter, ω and v

ω is a parameter introduced for a comparison with the traditional background-field method. When $\omega = 0$, it goes to harmonic gauge. Also a parameter, v, is introduced for the connection terms,

$$\nabla_i \nabla_j S[\bar{\varphi}] = S_{,ij}[\bar{\varphi}] - v\Gamma^k_{ij} S_{,k}[\bar{\varphi}] \tag{27}$$

At one-loop order with Landau-DeWitt gauge, the effective action is

$$\begin{aligned} \Gamma[\bar{\varphi}] &= S[\bar{\varphi}] - \ln\det Q_{\alpha\beta}[\bar{\varphi}] \\ &+ \frac{1}{2}\lim_{\Omega\to 0} \ln\det\left(\nabla^i \nabla_j S[\bar{\varphi}] + \frac{1}{2\Omega} K^i_\alpha[\bar{\varphi}] K^\alpha_j[\bar{\varphi}]\right). \end{aligned} \tag{28}$$

Use

$$\mathcal{I}^R_{2\mu\nu} = a_2 g_{\mu\nu} \mathcal{I}^R_2 \tag{29}$$

One-loop correction

The total quadratically divergent one-loop gravitational contribution to the effective action is given

$$\hat{\Gamma}[\bar{\mathrm{A}}_\mu] \quad = \quad \frac{1}{4}\int \mathrm{d}^4\mathrm{x}\bar{\mathcal{F}}^2 + \kappa^2\mathrm{C}\mathcal{I}_2^{\mathrm{R}}\frac{1}{4}\int \mathrm{d}^4\mathrm{x}\bar{\mathcal{F}}^2 \tag{30}$$

where the constant C is given by

$$\begin{aligned} \mathrm{C} \quad = \quad & \frac{4\mathrm{a}_2 - 1}{8}\Big(\mathrm{v}\left[(2\zeta - 1) - 4(\kappa^2\xi - 1)\right] \\ & + 8(\kappa^2\xi - 1) - 16\omega - 4\Big) + \omega(-1 + 6\mathrm{a}_2) \end{aligned} \tag{31}$$

obtain the gravitational correction to the β function

$$\beta_{\mathrm{g}}^{\kappa} = \frac{1}{2}\mathrm{g}\mu\frac{\partial}{\partial\mu}\delta_{\mathrm{A}}, \qquad \delta_{\mathrm{A}} \simeq -\kappa^2\mathrm{C}\mathcal{I}_2^{\mathrm{R}} \tag{32}$$

Results

Note that we used the notation for the quadratic divergences,

$$\mathcal{I}^{\mathrm{R}}_{2\mu\nu} = \mathrm{a}_2 \mathrm{g}_{\mu\nu} \mathcal{I}^{\mathrm{R}}_2. \tag{33}$$

For different regularization schemes, a_2 is different. We list the results

C	$\mathrm{a}_2 = \frac{1}{4}$	$\mathrm{a}_2 = \frac{1}{2}$
$\mathrm{v} = 1, \omega = 1,$ $\zeta \to 0,\ \ \xi \to 0$	$\frac{1}{2}$	$-\frac{9}{8}$
$\mathrm{v} = 0, \omega = 0,$ $\zeta = 1/2, \xi = 1/\kappa^2$	0	$-\frac{1}{2}$

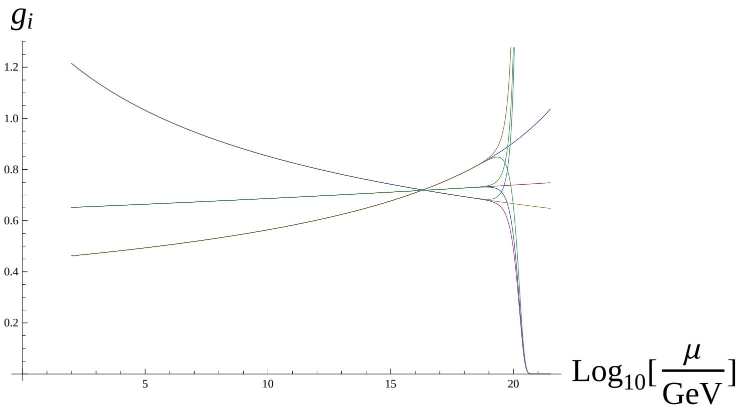


Figure: An illustration of gravitational contributions to the running of gauge couplings in the MSSM

Summary

- We considered the gravitational contributions to the runnning of gauge couplings.
- Both the traditional Diagrammatical and Background Field Method give the same result that the gravity tends to render all gauge theories asymptotically free. However the result is gauge condition dependent.
- We present the gauge condition independent result in the framework of the Vilkovisky-DeWitt formalism.

THANK YOU!