# Phase structure and critical behavior of black holes and branes in canonical ensemble

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# Introduction

Bekenstein (72) and Hawking (74) established:

- a stationary black hole with $M\,, Q \,\&\, J$ is a thermodynamical system,
- obeying the usual thermodynamical law such as the first law

$$dM = TdS + \Omega dJ + \Phi dQ, \tag{1.1}$$

where $k = \hbar = c = G = 1$.

# Introduction

In particular, a black hole has a temperature given by

$$T_{\rm BH} = \frac{\kappa}{2\pi} \quad \left( = \frac{\hbar c \kappa}{2\pi k} \right), \tag{1.2}$$

with $\kappa$ the so-called surface gravity of horizon, and an entropy given by

$$S = \frac{A}{4} \quad \left( = \frac{k c^3 A}{4 G \hbar} \right), \tag{1.3}$$

with $A$ the area of the horizon.

$\Rightarrow$ Quantum Thermodynamics?

$\Rightarrow$ Part of Quantum Gravity?

# Introduction

While with the above a black hole appears as a well-defined thermodynamical system, there exists a serious issue for asymptotically flat black hole with such an interpretation.

For example, a Schwarzschild black,

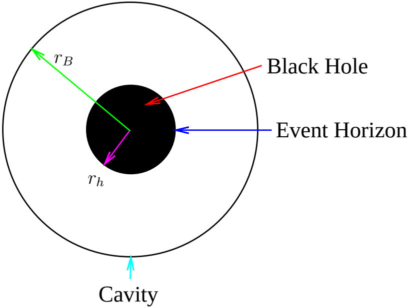
$$S_{\rm BH} = 4\pi M^2, \qquad T_{\rm BH} = \frac{1}{8\pi M} \tag{1.4}$$

with $M$ the ADM energy carried by the black hole. This system is actually thermodynamically unstable (the specific heat $C < 0$)!!!

# Introduction

- So in order to give a proper consideration of asymptotically flat black hole thermodynamics, we need first to suitably stabilize the black hole thermally.
- In other words, we need to consider ensembles that include not only the black hole under consideration but also its environment.
- Further, as self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest but also the place at which they take specific values.

# Introduction




Black hole ($r_h$) placed in a cavity ($r_B$) with fixed T and V.

# Chargeless case

Consider the simplest spherical symmetric Schwarzschild black hole in Euclidean signature

$$ds_E^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2, \qquad (2.1)$$

with the horizon radius $r_h = 2M$. If we place this black hole in a large spherical hot cavity at a given $r = r_B$ ($r_B > r_h$) with the temperature at the wall fixed at $T$, this will define a canonical ensemble *a la* York (PRD33 (1986) 2092) for this hole. Thermal equilibrium says

$$T = T_{BH}(r_B) = T_{BH}\left(1 - \frac{2M}{r_B}\right)^{-1/2} = (8\pi M)^{-1}\left(1 - \frac{2M}{r_B}\right)^{-1/2} \qquad (2.2)$$

# Chargeless case

The stability can be analyzed using the Helmholz free energy which can be calculated following Gibbons and Hawking (PRD15(1977)2752) that the partition function $Z$ contains the first-order classical Euclidean Einstein action of the hole as its leading term.

In other words,

$$Z = e^{-\beta F} \approx e^{-I_E} \tag{2.3}$$

$\Rightarrow$

$$I_E(r_B, T; r_h) = \beta F = \beta E(r_B, T; r_h) - S(r_h) \tag{2.4}$$

with $\beta = 1/T$ and $E$ the internal energy of the cavity.

# Chargeless case

$$
\begin{aligned}
E(r_B; x) &= r_B \left(1 - (1-x)^{1/2}\right), \\
S(r_B; x) &= \pi r_B^2 x^2 = 4\pi M^2 \qquad (2.5)
\end{aligned}
$$

with

$$x \equiv \frac{r_h}{r_B} = \frac{2M}{r_B}, \qquad r_B > r_h \qquad (2.6)$$

Note

$$0 < x < 1. \qquad (2.7)$$

$$I_E = \beta r_B \left(1 - (1-x)^{1/2}\right) - \pi r_B^2 x^2, \qquad (2.8)$$

# Chargeless case

For simplicity, define

$$\bar{I}_E \equiv \frac{I_E}{4\pi r_B^2}, \quad \bar{b} \equiv \frac{\beta}{4\pi r_B}, \tag{2.9}$$

$$\bar{I}_E = \bar{b}\left(1-(1-x)^{1/2}\right) - \frac{1}{4}x^2. \tag{2.10}$$

$$\left(\bar{I}_E = 0\,(x=0) \qquad \Leftrightarrow \qquad \text{hot flat space}\right). \tag{2.11}$$

$$\frac{\partial \bar{I}_E}{\partial x} = \frac{1}{2(1-x)^{1/2}}\left(\bar{b} - b(x)\right), \tag{2.12}$$

$$b(x) = x(1-x)^{1/2} > 0. \tag{2.13}$$

Note

$$b(x\to 0)\to 0, \qquad b(x\to 1)\to 0. \tag{2.14}$$

# Chargeless case

$$\frac{\partial \bar{I}_E}{\partial x} = 0 \Rightarrow \bar{b} = b(\bar{x}) = \bar{x}(1-\bar{x})^{1/2} \tag{2.15}$$

$$\Rightarrow T = T(r_B) = (8\pi M)^{-1}\left(1 - \frac{2M}{r_B}\right)^{-1/2}. \tag{2.16}$$

# Chargeless case

$$\left.\frac{\partial^2 \bar{I}_E}{\partial x^2}\right|_{x=\bar{x}} \sim -\frac{\partial b(\bar{x})}{\partial \bar{x}}, \tag{2.17}$$

$$\begin{aligned} &\frac{\partial b(x)}{\partial x} > 0, \qquad \frac{\partial^2 \bar{I}_E}{\partial x^2} < 0 \qquad (\text{unstable}) \\ &\frac{\partial b(x)}{\partial x} < 0, \qquad \frac{\partial^2 \bar{I}_E}{\partial x^2} > 0 \qquad (\text{stable}) \end{aligned} \tag{2.18}$$

# Chargeless case

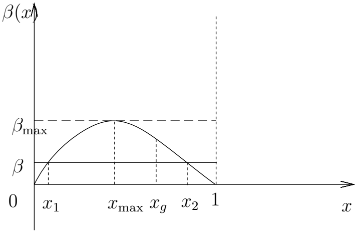



Figure 1: The typical behavior of $\beta(x)$ vs $x$ ($x \equiv r_h/r$).

# Charged case

The charged (Reissner-Nordström) black hole is

$$ds_E^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_2^2, \tag{3.1}$$

with

$$V(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}, \qquad \Phi = \frac{e}{r}. \tag{3.2}$$

It has two horizons given at (V (r) = 0)

$$r_\pm = M \pm \sqrt{M^2 - e^2}, \tag{3.3}$$

which implies

$$M \geq e, \quad \text{(BPS Bound)} \tag{3.4}$$

## Charged case

By the same token,

$$\begin{aligned} I_E(\beta, r_B, e; r_+) &= \beta E(r_B, e; r_+) - S(r_+) \\ &= \beta\left(1 - \sqrt{\left(1 - \frac{r_+}{r_B}\right)\left(1 - \frac{e^2}{r_+ r_B}\right)}\right) - \pi r_+^2 \end{aligned} \tag{3.5}$$

Define,

$$\bar{I}_E \equiv \frac{I_E}{4\pi r_B^2}, \qquad x \equiv \frac{r_+}{r_B}, \quad q \equiv \frac{e}{r_B}, \quad \bar{b} \equiv \frac{\beta}{4\pi r_B}, \\ q < x < 1, \quad (r_+ > e, r_B > r_+). \tag{3.6}$$

# Charged case

$$\bar{I}_E(\bar{b}, q; x) = \bar{b}\left(1 - \sqrt{(1-x)\left(1 - \frac{q^2}{x}\right)}\right) - \frac{1}{4}x^2. \tag{3.7}$$

$$\frac{\partial \bar{I}_E}{\partial x} = \frac{1 - \frac{q^2}{x^2}}{2(1-x)^{1/2}\left(1 - \frac{q^2}{x}\right)^{1/2}}\left(\bar{b} - b_q(x)\right), \tag{3.8}$$

where

$$b_q(x) = \frac{x(1-x)^{1/2}\left(1 - \frac{q^2}{x}\right)^{1/2}}{1 - \frac{q^2}{x^2}}. \tag{3.9}$$

Note

$$b_q(x \to q) \to \infty, \quad b_q(x \to 1) \to 0. \tag{3.10}$$

## Charged case

$$\frac{\partial \bar{I}_E}{\partial x} = 0 \qquad \Rightarrow \qquad \bar{b} = b_q(\bar{x}). \tag{3.11}$$

$$\Downarrow$$

$$T = T(r_B) = (4\pi r_+)^{-1}\left(1 - \frac{e^2}{r_+^2}\right)\left(1 - \frac{r_+}{r_B}\right)^{-1/2}\left(1 - \frac{e^2}{r_+ r_B}\right)^{-1/2}. \tag{3.12}$$

# Charged case

Once again,

$$\left.\frac{\partial^2 \bar{I}_E}{\partial x^2}\right|_{x=\bar{x}} \sim -\frac{\partial b(\bar{x})}{\partial \bar{x}}, \tag{3.13}$$

The locally stable black hole requires

$$\frac{\partial b(\bar{x})}{\partial \bar{x}} < 0. \tag{3.14}$$

# Charged case

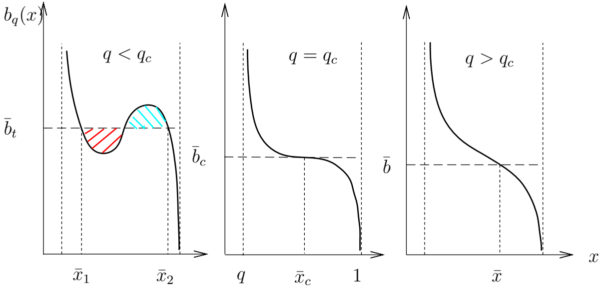
Note that

$$b_q(x \to q) \to \infty, \quad b_q(x \to 1) \to 0, \tag{3.15}$$

there exists a critical charge $q_c = \sqrt{5} - 2(\bar{x}_c = 5 - 2\sqrt{5}, \bar{b}_c = 0.429)$ and we actually have three cases to consider:

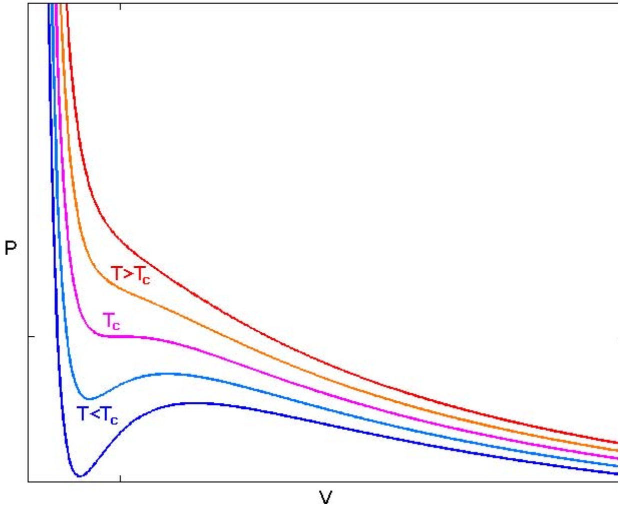
- $q < q_c$, there exists a unique temperature $T_t(q)$ for each given $q$ at which there exists a first order phase transition between a small and large black holes. We have a line of this first-order phase transition, depending on $q < q_c$ and ending at a second-order phase transition point at $q = q_c$;
- $q = q_c$, this is a second-order critical point at which there exists no distinction between small and large black holes. The critical exponent can be read from $c_v \sim (T - T_c)^{-2/3}$ as $-2/3$;
- $q > q_c$ for each given temperature $T$ there exists a unique global stable black hole with size $r_+ = r_B \bar{x}$.

# Charged case




The typical behaviors of $b(x)$ vs $x$ for $q < q_c, q = q_c, q > q_c$.

# Van der Waals isotherm



So much for the usual black holes!

In string/M-theory, the basic objects are the so-called p-branes and the black correspondences are the asymptotically flat black p-branes with each having a horizon.

Then what happen to the thermodynamical behavior of these branes and the phase structure? (Lu et al JHEP 1101:133(2011))

# p-brane

A p-brane is a p-dimensional hyperspace $(p = 0, 1, \cdots, 9)$ residing at the bulk spacetime with dimension $D$ ( $D \geq p + 1)$ and can carry either electric-like $d + 1$-form charge with $d = p + 1$ as

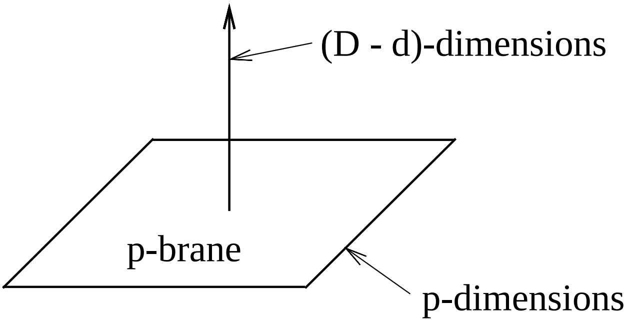
$$e_d \sim \int {}^*F_{d+1} \tag{4.1}$$

or magnetic-like $\tilde{d} + 1$-form charge with $\tilde{d} = D - 2 - d$ as

$$g_{\tilde{d}} \sim \int F_{\tilde{d}+1}. \tag{4.2}$$

# p-brane

The spatial dimensions transverse to the p-brane is $D - d = \tilde{d} + 2$ and note $1 \leq \tilde{d} \leq 7$.



# Black p-brane configuration

The black -brane configuration in Euclidean signature is

$$
\begin{aligned}
ds_E^2 &= \Delta_+ \Delta_-^{-\frac{d}{D-2}} dt^2 + \Delta_-^{\frac{\tilde d}{D-2}} \sum_{i=1}^{d-1} (dx^i)^2 + \Delta_+^{-1} \Delta_-^{\frac{a^2}{2\tilde d}-1} d\rho^2 \\
&\quad + \rho^2 \Delta_-^{\frac{a^2}{2\tilde d}} d\Omega_{\tilde d+1}^2, \\
A_{[p+1]} &= -i e^{a\phi_0/2} \left[ \left(\frac{r_-}{r_+}\right)^{\tilde d/2} - \left(\frac{r_- r_+}{\rho^2}\right)^{\tilde d/2} \right] dt \wedge dx^1 \wedge \ldots \wedge dx^p, \\
F_{[p+2]} &\equiv dA_{[p+1]} = -i e^{a\phi_0/2} \tilde d \, \frac{(r_- r_+)^{\tilde d/2}}{\rho^{\tilde d+1}} d\rho \wedge dt \wedge dx^1 \wedge \ldots \wedge dx^p, \\
e^{2(\phi-\phi_0)} &= \Delta_-^a,
\end{aligned}
\tag{4.3}
$$

where

$$
\Delta_\pm = 1 - \frac{r_\pm^{\tilde d}}{\rho^{\tilde d}}. \tag{4.4}
$$

## The equation of state

The present equation of state is

$$\bar{b} = b_q(x) \equiv \frac{x^{1/\tilde{d}}(1-x)^{1/2}}{\tilde{d}\left(1-\frac{q^2}{x^2}\right)^{\frac{1}{2}-\frac{1}{\tilde{d}}}\left(1-\frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}, \qquad (4.5)$$

Note

$$b_q(x \to q) \to \infty(\tilde{d} > 2), \quad b_q(x \to 1) \to 0 \qquad (4.6)$$

# The reduced action

and the reduced action

$$\bar{I}_E(\bar{b}, q; x) \equiv \frac{2\kappa^2 I_E}{4\pi\bar{\rho}_B^{\tilde{d}+1} V_p \Omega_{\tilde{d}+1}}$$

$$= -\bar{b}\left[(\tilde{d}+2)\left(\frac{1-x}{1-\frac{q^2}{x}}\right)^{1/2} + \tilde{d}(1-x)^{1/2}\left(1-\frac{q^2}{x}\right)^{1/2}\right.$$

$$\left. -2(\tilde{d}+1)\right] - x^{1+\frac{1}{\tilde{d}}}\left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}}. \quad (4.7)$$

# Phase structure and transition

For each given $\tilde{d} > 2$, the phase structure here is the same as the charged black hole though the detail is different. There exists a critical charge $q_c$, depending on $\tilde{d}$, and we have also the three cases for each given $\tilde{d} > 2$,

# Critical quantities and exponents

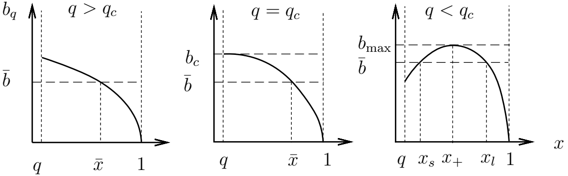
The relevant quantities at the critical point can be calculated explicitly for each allowed value of $\tilde{d}$ as:

| $\tilde{d}$ | $q_c$ | $x_c$ | $b_c$ |
|---|---|---|---|
| 3 | 0.141626 | 0.292656 | 0.199253 |
| 4 | 0.090672 | 0.238800 | 0.159921 |
| 5 | 0.064944 | 0.202012 | 0.134632 |
| 6 | 0.049599 | 0.175176 | 0.116698 |
| 7 | 0.039529 | 0.154691 | 0.103210 |

The critical exponents $\alpha$ of $c_v \sim (T - T_c)^\alpha$ can be calculated straightforward and take a universal value of $-2/3$, independent of $\tilde{d}$.

# Phase structure

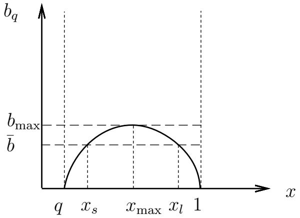
The $\tilde{d} = 2$ case $(q_c = 1/3)$:




The typical behaviors of $b_q(x)$ vs $x$ for $\tilde{d} = 2$.

# Phase structure

The $\tilde{d} = 1$ case:




The typical behavior of $b_q(x)$ vs $x$ for $\tilde{d} = 1$.

# The conclusion

## Conclusion

- We have found van der Waals-Maxwell like phase structure and transition for black p-branes in canonical ensemble when $q < q_c$.
- There exists a first order phase transition line when the charge moves from $q < q_c$ towards $q = q_c$, ending up at a second order phase transition point (critical point) when $q = q_c$.
- We calculated explicitly the critical quantities $(q_c, x_c, b_c)$ and found that they all decrease when $\tilde{d}$ increases. The critical exponent is calculated to be $-2/3$, independent of $\tilde{d}$, in this ensemble.
- We also found that branes with the same value of $\tilde{d}$ share the same phase behavior at least to the leading approximation employed in this work.

# THANK YOU!