Spinor form of the amplitude with massive field

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Outline

1. Introduction
2. Three point amplitude for the general massive fields
3. Recursion relations for the amplitude with massive fields
4. Future plan

Motivation

- ★ Constrain the interaction form (three point function) for the fields of general spin
- ★ Calculation the amplitude with massive fields more effectively
- ★ Transformed to the twistor space to see the algebra geometry curves

Amplitude in spinor form

General form of the amplitude

$A(\lambda_1, \tilde{\lambda}_1, \beta_1, \tilde{\beta}_1, \psi_1^{s_z}; \cdots)$

- Linear property with respect to external fields
- Lorenz invariant
- momentum conservation
- Functions of spinors

Spinor form for the momentum and external fields –($s = \frac{1}{2}$)

Massless field $p^2 = 0$

Spinor form $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

Massive fields $p^2 = m^2$

$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \beta_a \tilde{\beta}_{\dot{a}}$

The solution of E.O.M (dirac equation)

Massless	Left-hand:λ, right-hand:$\tilde{\lambda}$
Massive	$u^- = \binom{\lambda}{\tilde{\beta}}, \quad u^+ = \binom{\beta}{-\tilde{\lambda}}, \quad v^- = \binom{\lambda}{-\tilde{\beta}}, \quad v^+ = \binom{\beta}{\tilde{\lambda}}.$

Generator of the little group in the spinor form

For massive fields ,the little group is $SO(3)$, it can be represented as the first-order differential operators with respect to spinors

$$\begin{aligned}
\mathscr{R}(J^1) &= \frac{-1}{2}\left(\beta\frac{\partial}{\partial\lambda}-\tilde{\beta}\frac{\partial}{\partial\tilde{\lambda}}+\lambda\frac{\partial}{\partial\beta}-\tilde{\lambda}\frac{\partial}{\partial\tilde{\beta}}\right), \\
\mathscr{R}(J^2) &= \frac{i}{2}\left(\beta\frac{\partial}{\partial\lambda}+\tilde{\beta}\frac{\partial}{\partial\tilde{\lambda}}-\lambda\frac{\partial}{\partial\beta}-\tilde{\lambda}\frac{\partial}{\partial\tilde{\beta}}\right), \\
\mathscr{R}(J^3) &= \frac{-1}{2}\left(\lambda\frac{\partial}{\partial\lambda}-\tilde{\lambda}\frac{\partial}{\partial\tilde{\lambda}}-\beta\frac{\partial}{\partial\beta}+\tilde{\beta}\frac{\partial}{\partial\tilde{\beta}}\right).
\end{aligned} \tag{1}$$

Check the commutation relations

$$[\mathscr{R}(J^i),\mathscr{R}(J^j)] = i\varepsilon_{ijk}\mathscr{R}(J^k). \tag{2}$$

External field forms a representation of the little group

$$\begin{aligned}
\mathscr{R}(J^3)u_s = \frac{-1}{2}\left(\lambda\frac{\partial}{\partial\lambda} - \tilde{\lambda}\frac{\partial}{\partial\tilde{\lambda}} - \beta\frac{\partial}{\partial\beta} + \tilde{\beta}\frac{\partial}{\partial\tilde{\beta}}\right)u_s &= \mathscr{D}^{\frac{1}{2}}_{rs}(J^3)u_r, \\
\mathscr{R}(J^2)u_s = \frac{+i}{2}\left(\beta\frac{\partial}{\partial\lambda} + \tilde{\beta}\frac{\partial}{\partial\tilde{\lambda}} - \lambda\frac{\partial}{\partial\beta} - \tilde{\lambda}\frac{\partial}{\partial\tilde{\beta}}\right)u_s &= \mathscr{D}^{\frac{1}{2}}_{rs}(J^2)u_r, \\
\mathscr{R}(J^1)u_s = \frac{-1}{2}\left(\beta\frac{\partial}{\partial\lambda} - \tilde{\beta}\frac{\partial}{\partial\tilde{\lambda}} + \lambda\frac{\partial}{\partial\beta} - \tilde{\lambda}\frac{\partial}{\partial\tilde{\beta}}\right)u_s &= \mathscr{D}^{\frac{1}{2}}_{rs}(J^1)u_r.
\end{aligned}$$

the raising and lowering operators

$\mathscr{R}(J^+) = \left(\tilde{\lambda}\frac{\partial}{\partial\tilde{\beta}} - \beta\frac{\partial}{\partial\lambda}\right)$ and $\mathscr{R}(J^-) = \left(\tilde{\beta}\frac{\partial}{\partial\tilde{\lambda}} - \lambda\frac{\partial}{\partial\beta}\right)$

Application on $\frac{3}{2}$ spin fields

The states for spin $\frac{3}{2}$

$$
\begin{aligned}
u_{\frac{-3}{2}} &= -\lambda\tilde{\beta}(\lambda \oplus \tilde{\beta}) \\
u_{\frac{-1}{2}} &= (-\lambda\tilde{\lambda} + \beta\tilde{\beta})(\lambda \oplus \tilde{\beta}) + \lambda\tilde{\beta}(\beta \oplus -\tilde{\lambda}) \\
u_{\frac{1}{2}} &= (-\lambda\tilde{\lambda} + \beta\tilde{\beta})(\beta \oplus -\tilde{\lambda}) - \beta\tilde{\lambda}(\lambda \oplus \tilde{\beta}) \\
u_{\frac{3}{2}} &= -\beta\tilde{\lambda}(\beta \oplus -\tilde{\lambda})
\end{aligned} \tag{3}
$$

Acting the $\mathscr{R}(J^{+})$ on the $u_{\frac{-3}{2}}$

Similarly, for anti-particles

$$
\begin{aligned}
v_{\frac{-3}{2}} &= -\lambda\tilde{\beta}(\lambda \oplus -\tilde{\beta}) \\
v_{\frac{-1}{2}} &= (-\lambda\tilde{\lambda} + \beta\tilde{\beta})(\lambda \oplus -\tilde{\beta}) + \lambda\tilde{\beta}(\beta \oplus \tilde{\lambda}) \\
v_{\frac{1}{2}} &= (-\lambda\tilde{\lambda} + \beta\tilde{\beta})(\beta \oplus \tilde{\lambda}) - \beta\tilde{\lambda}(\lambda \oplus -\tilde{\beta}) \\
v_{\frac{3}{2}} &= -\beta\tilde{\lambda}(\beta \oplus \tilde{\lambda}).
\end{aligned} \tag{4}
$$

Apply to the general Rarita-Schwinger tensor spinors

The lowest helicity vector is obtained by symmetry product the tensor part and the dirac-spinor part as

$$u_{-j} = -(\lambda\tilde{\beta})^{j-\frac{1}{2}}(\lambda \oplus \tilde{\beta}). \tag{5}$$

The the higher helicity states can be obtained as

$$u_{(-j+n)} = N(j,n)(J^{+})^{n}u_{-j}, \tag{6}$$

where n in integer and $\in [1, 2j]$ and N is the normalize constant.

Manifest forms

$$
\begin{aligned}
& u_{-j+n} \\
= & \sum_{i=0}^{[\frac{n}{2}]} \mathscr{S}\left((\lambda\tilde{\beta})^{\otimes j-\frac{1}{2}+i-n} \otimes (\lambda\tilde{\lambda}-\beta\tilde{\beta})^{\otimes(n-2i)}(-2\beta\tilde{\lambda})^{\otimes i}\right) \otimes \lambda \\
- & \sum_{i=0}^{[\frac{n-1}{2}]} \mathscr{S}\left((\lambda\tilde{\beta})^{\otimes j-\frac{1}{2}+i+1-n} \otimes (\lambda\tilde{\lambda}-\beta\tilde{\beta})^{\otimes(n-2i-1)}(-2\beta\tilde{\lambda})^{\otimes i}\right) \otimes \beta .
\end{aligned}
$$

$\mathscr{S}$ denote the symmetry tensor product

Amplitude in spinor form

We consider the three point amplitude with ($\bar{q}$ q g). According to the function property of the amplitude in the spinor form, the three point amplitude should satisfy the following one order differential equations

$$\begin{aligned}
\mathscr{R}(J^a)_1 A(\lambda_i,\tilde{\lambda}_i,\beta_i,\tilde{\beta}_i,u^i_{r_i};\lambda_3,\tilde{\lambda}_3,\pm) &= A(\lambda_i,\tilde{\lambda}_i,\beta_i,\tilde{\beta}_i,\mathscr{D}^{(2)}_{sr_1}u^1_s,u^2_{r_2};\lambda_3,\tilde{\lambda}_3, \\
\mathscr{R}(J^a)_2 A(\lambda_i,\tilde{\lambda}_i,\beta_i,\tilde{\beta}_i,u^i_{r_i};\lambda_3,\tilde{\lambda}_3,\pm) &= A(\lambda_i,\tilde{\lambda}_i,\beta_i,\tilde{\beta}_i,\mathscr{D}^{(2)}_{sr_2}u^2_s,u^1_{r_1},\lambda_3,\tilde{\lambda}_3, \\
\mathscr{R}(J^3)_3 A(\lambda_i,\tilde{\lambda}_i,\beta_i,\tilde{\beta}_i,u^{r_i}_i,\lambda_3,\tilde{\lambda}_3,\pm) &= \pm A(\lambda_i,\tilde{\lambda}_i,\beta_i,\tilde{\beta}_i,u^{r_1}_1,u^{r_2}_2;\lambda_3,\tilde{\lambda}_3,\pm
\end{aligned}$$

Relation: 7; Constraint: 4, Freedom: 10

It's hard to solve these directly. (One of the possible method is to use the $\mathscr{D}$-module technology in algebra geometry)

Directly construction of the solutions.

According to the general properties of the amplitude, we construct the solutions directly.

The general form of the amplitudes for spin $\frac{1}{2}$

$$
\begin{aligned}
A = \quad (\quad & c_1 \langle\lambda_1|F_1^e(p_1,p_2)|\lambda_2\rangle \; [\tilde{\mu}|F_2^o(p_1,p_2)|\lambda_3\rangle \\
+ \quad & c_2 \langle\lambda_1|F_3^o(p_1+p_2)|\tilde{\mu}] \; \langle\lambda_2,\lambda_3\rangle \\
+ \quad & c_3 \langle\lambda_2|F_4^o(p_1+p_2)|\tilde{\mu}] \; \langle\lambda_1,\lambda_3\rangle \\
+ \quad & c_4 \langle\lambda_1|F_5^o(p_1)|\mu] \; \langle\lambda_2,\lambda_3\rangle + c_{4'} \langle\lambda_2|F_{5'}^o(p_2)|\mu] \langle\lambda_1,\lambda_3\rangle \Big) \\
& \frac{1}{[\tilde{\mu},\tilde{\lambda}_3]}
\end{aligned} \tag{8}
$$

General solutions

Only three independent solutions and the general form of the amplitude is

$$A(\frac{-1}{2},\frac{-1}{2},-1)=\frac{P^1(\langle\lambda_1,\lambda_2\rangle,[\tilde{\beta}_1,\tilde{\beta}_2])\langle\lambda_2,\lambda_3\rangle}{[\tilde{\beta}_2,\tilde{\lambda}_3]}+Q^0\langle\lambda_1,\lambda_3\rangle\,\langle\lambda_2,\lambda_3\rangle. \tag{9}$$

Here Q^0 only depends on the mass and P^1 is a first-order polynomial function: $a(m)\langle\lambda_1,\lambda_2\rangle+b(m)[\tilde{\beta}_1,\tilde{\beta}_2]$,
$b(m)$ term $\leftrightarrow$ the electric interactions
Q^0 term $\leftrightarrow$ magnetic interactions
The term of coefficient $a(m)$ is not consistent with the gauge invariance when the external photon are off shell.

Amplitude with other spin-configurations

According to the spinor form of the little group generators, we can directly obtain the other amplitudes

$$\begin{aligned}
A(\frac{1}{2},\frac{-1}{2},-1) &= \mathscr{R}(J_1^+)A(\frac{-1}{2},\frac{-1}{2},-1) \\
A(\frac{-1}{2},\frac{1}{2},-1) &= \mathscr{R}(J_2^+)A(\frac{-1}{2},\frac{-1}{2},-1) \\
A(\frac{1}{2},\frac{1}{2},-1) &= \mathscr{R}(J_1^+)\mathscr{R}(J_2^+)A(\frac{-1}{2},\frac{-1}{2},-1) \qquad (10)
\end{aligned}$$

General solutions for other spins

For $\frac{3}{2}$ spin

$$A\Big((\frac{3}{2},-\frac{3}{2}),(\frac{3}{2},-\frac{3}{2}),(1,-1)\Big) = \frac{P^3\left([\tilde{\beta}_1,\tilde{\beta}_2],\langle\lambda_1,\lambda_2\rangle\right)\langle\lambda_3,\lambda_2\rangle}{[\tilde{\beta}_2,\tilde{\lambda}_3]}$$
$$+ \quad Q^2\left([\tilde{\beta}_1,\tilde{\beta}_2],\langle\lambda_1,\lambda_2\rangle\right)\langle\lambda_1,\lambda_3\rangle\langle\lambda_3,\lambda_2\rangle, \qquad (11)$$

For general spin

$$A\Big((j,-j),(j,-j),(1,-1)\Big) = \frac{F^{2j}\left([\tilde{\beta}_1,\tilde{\beta}_2],\langle\lambda_1,\lambda_2\rangle\right)\langle\lambda_3,\lambda_2\rangle}{[\tilde{\beta}_2,\tilde{\lambda}_3]}$$
$$+ \quad G^{2j-1}\left([\tilde{\beta}_1,\tilde{\beta}_2],\langle\lambda_1,\lambda_2\rangle\right)\langle\lambda_1,\lambda_3\rangle\langle\lambda_3,\lambda_2\rangle, \qquad (12)$$

General review of the BCFW recursion relations

Such relation is accomplished by shifting the external momentum $p(z)$ such that the external fields are still on shell and the momentum conservation hold. The amplitude with shifted external momentum are denoted as $\mathscr{A}(z)$.

Three conditions for the recursion relations $cite(BCFW:2005)$

- Rational condition: $\mathscr{A}(z)$ is rational
- Constructive condition: it vanish for $z \to \infty$
- Simple pole condition: the only singularities are simple poles

If the conditions are met, there are simple recursion relations for these constructive amplitudes (citeBadger1,Badger2)

$$\mathscr{A}(p_1,\cdots p_n)=\sum_{pt}\sum_{h}\mathscr{A}_L(p_r,\cdots\hat{p}_i,\cdots,p_s,-\hat{P}^h)$$

$$\frac{1}{P_{ij}^2-m_{P_ij}^2}\mathscr{A}_R(\hat{P}^h,p_{s+1},\cdots\hat{p}_j,\cdots,p_{r-1}), \qquad (13)$$

Progress on the tree-level amplitude

- ★ Amplitude with more than one massless external line (Solved by S. D. Badger et.al. [arXiv: hep-th/0504159], [arXiv: hep-th/0507161] and also in Ozeren's [arXiv: hep-ph/0603071], Schwinn's [arXiv: 0809.1442]
- ★ Amplitude with only one massless external line (Solved in arXiv:1103.2518)
- ★ Amplitude with all massive lines (Solved in arXiv:1103.2518)

Major obstruct

Non-constructible: $\mathscr{A}(z)_{z\to\infty} \neq 0$. If the shifting lines contain massive lines, it it is impossible for the amplitude to be constructible for general spin configurations. Fortunately, we find it is possible to choose a correlated spin configurations for the two shift line such that the amplitude is constructible.

Massive momentum shifting scheme

Since the amplitudes are Lorentz invariant, we can choose a reference frame such that the two shifted momentum can be of form

$$\begin{aligned} p_{\hat{q}_1} &= \lambda_{q_1}\tilde{\lambda}_{q_1} + \beta_{q_1}\tilde{\beta}_{q_1} + z\lambda_{q_1}\tilde{\beta}_{q_1}, \\ p_{\hat{q}_2} &= \lambda_{q_2}\tilde{\lambda}_{q_2} + \beta_{q_2}\tilde{\beta}_{q_2} - z\lambda_{q_1}\tilde{\beta}_{q_1}, \end{aligned} \tag{14}$$

and

$$\langle \lambda_{q_1}, \lambda_{q_2}\rangle [\tilde{\beta}_{q_1}, \tilde{\lambda}_{q_2}] + \langle \lambda_{q_1}, \beta_{q_2}\rangle [\tilde{\beta}_{q_1}, \tilde{\beta}_{q_2}] = 0. \tag{15}$$

The z-independent states for two labeled particles are

$$\begin{pmatrix} \lambda_{q_1} \\ \tilde{\beta}_{q_1} \end{pmatrix}, \begin{pmatrix} a \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} -p_{q_2} \circ \tilde{\beta}_{q_1} \\ m\tilde{\beta}_{q_1} \end{pmatrix} \tag{16}$$

momentum shifting scheme for one massive and one massless lines

For massless gauge field of $+$ helicity, the amplitudes are constructible under the two-line shift (cite: Badger1, Badger2)

$$
\begin{aligned}
p_1 &= \lambda_1\tilde{\lambda}_1 + z([\tilde{\lambda}_2,\tilde{\lambda}_1]\lambda_2\tilde{\lambda}_1 + [\tilde{\beta}_2,\tilde{\lambda}_1]\beta_2\tilde{\lambda}_1), \\
p_2 &= \lambda_2\tilde{\lambda}_2 + \beta_2\tilde{\beta}_2 - z([\tilde{\lambda}_2,\tilde{\lambda}_1]\lambda_2\tilde{\lambda}_1 + [\tilde{\beta}_2,\tilde{\lambda}_1]\beta_2\tilde{\lambda}_1), \quad (17)
\end{aligned}
$$

where p_1 momentum of the massless gauge boson and p_2 is for the dirac fields.

External field for the constructible amplitude

$\varepsilon^+ = \frac{\mu\tilde{\lambda}_1}{\langle \mu,\lambda_1\rangle}$, $\binom{a}{\tilde{b}} = \binom{-m\lambda_1}{\widetilde{\lambda_1 \circ p_q}}$

The amplitude with spin state independent with each other

Here, the spin state of the quark line are not independent but related with the spinors of a gluon $\binom{a}{\tilde{b}} = \binom{-m\lambda_1}{\widetilde{\lambda_1 \circ p_q}}$.
To get the amplitude with another spin state for this massive line l_c, we can shift the momentum of this line together with another massive line l_f, $\binom{a}{\tilde{b}} = \binom{-p_{q_2} \circ \tilde{\beta}_{q_1}}{m\tilde{\beta}_{q_1}}$.
Linear combination of the two parts, we get the amplitude with non-related spin states.

General procedure for the tree-level amplitude in QCD

1. Choose two massless line to be shifted if possible.
2. If not, choose two pair of massive lines to be shifted. The two pair of lines have a common fields.
3. Recursion the amplitude of two different spin configuration for the non-common fields respectively to be of less external lines for the two pair of momentum shifting.
4. Linear combine the two spin configurations of the amplitude and obtain an amplitude which has lines of independent spin state .
5. Acting with spinor form of the little group generators, to obtain the amplitude of arbitrary spin configuration.

Future plan

- ★ Extended to one loop
- ★ Extended to higher spin fields
- ★ Transform to the twistor space, to analysis the algebra curve corresponding to each non-vanishing amplitude
- ★ Apply to a system of modular space (e.x. N=2 SUSY theory). And discuss the transformation manner of the amplitude under the moving in modular space.

Thank you