1. Theory of Special relativity

1.1 Fund. Postulates of SR

Special relativity was proposed by Einstein in 1905 to solve a number of experimental results that were hard to explain with the theoretical understanding at that time.

SR can be derived from the following two postulates:

1° The law of nature and the results of all experiments performed in a given frame of reference are indep. of the uniform translational motion of the system as a whole.
(Galileo principle of relativity)

2° Constancy of the speed of light (Einstein)

The velocity of light is independent of the uniform translational motion of the source.

In more explicit terms,

two different unaccelerated observers measuring the speed of light will find it to be the same constant $c = 2.99792 \times 10^8$ m/s

uniform translational motion means a motion with constant velocity.

- Historically, Newton's law was found to be invariant under the transformation

$$\begin{cases} x' \to x - vt \\ t' \to t \end{cases} \quad \text{called Galileo transformation.} \qquad (1.1)$$

This transformation led to the concept of absolute space & absolute time.

universal, exists indep of observers

But Maxwell eqn was found to be incompatible with (1.1) since Maxwell eqn. implies the wave eqn.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \qquad (1.2)$$

with a constant $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. This wave eqn. asserts that there exists EM wave with with speed $v = c$. Now if Maxwell obeys postulate 1°, then it means (1.2) should hold for both observers S & S' where S' is moving with a constant. velocity u w.r.t. S.

ie. both S & S' will find the same EM wave signal moving at the same speed c.

ie. postulate 1° + Maxwell eqn. being a physical law of nature $\Rightarrow$ Constancy of Speed of light.

This ~~contradicts~~ the transformation (1.1)

Which is a consequence of postulate 1° and the assumption that Newton's law being a physical law of nature

Since it is very reasonable to have postulate 1°, the contradiction means we cannot have both Maxwell eqn & Newton eqn both hold as physical laws of nature at the same time!

Either: (i), Newton is correct, Maxwell is wrong

(ii), Maxwell is correct, Newton's law is wrong.

(iii), Both Newton's law and Maxwell eqns are wrong.

Before 1905, most physicists tried (i). For example, people tried to say Maxwell eqn holds only in a preferred frame where a hypothetical material called ether is at rest. However, the properties of ether needed to be very strange. Moreover, no experimental observation shows the existence of ether.

Einstein tried (ii) and proposed special relativity.

We will review SR in more details later.

Let us first clarify a few basic concepts.

1.2 Def. of an inertial observer in SR

An inertial frame is a coordinate system that satisfies the following properties:

i) The distance between point P_1 and point P_2 is indep. of time.

ii) The clocks that sit at every point of space ~~tick~~ are synchronized and run at the same rate.

iii) The geom. of space at any time t is Euclidean.

rmk:

1. Turns out only an unaccelerated observer can keep his clock synchronized.
2. The def is an ideal one. It is not self evident that such a frame exists in nature.
   In general, a grav. field makes it impossible to construct such a frame.
   However, in the presence of weak gravity, we can approximately realize such a frame.

3: An equiv def is to say that an inertial frame is defined such that the Maxwell eqn holds.

1.3 Natural units

In SI unit, distance is measured in terms of meter
time ——— sec.
velocity ——— ms^{-1}

However, since there is an universal speed c, it is natural to consider a unit such that $c = 1$ i.e. dimensionless.

In this unit, $v = 1/2$ (for example) means the speed is half that of the speed of light.

This is called the natural unit.

(Natural unit actually refers to a unit system such that $c = \hbar = 1$)

To convert from SI unit to the natural unit, we use relation like

$$3\times10^8\,ms^{-1} = 1$$

$$\Leftrightarrow \begin{cases} 1s = 3\times10^8\,m \\ 1m = \frac{1}{3\times10^8}\,s \end{cases} \quad \text{etc.}$$

1.4 Spacetime diagrams

The diagram of space (x_1, x_2, x_3) and time is called a spacetime diagram.

event = a single point in this space. ie (t, x_1, x_2, x_3)

$c = 1$

worldline = A line in this space $x_i = x_i(t)$.

Slope $= \frac{dt}{dx} = \frac{1}{v}$ $\quad$ v = Speed.

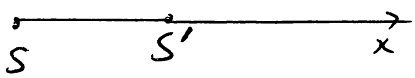
Convention: (t, x, y, z) or $(\underbrace{x^0, x^1, x^2, x^3}_{x^i})$ or x^μ $\quad$, $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$

Latin index

1.5 Construction of the coordinates used by another observer.

We will use the postulate of SR to ~~[illegible]~~ work out the relationship between the spacetime diag of S & S'

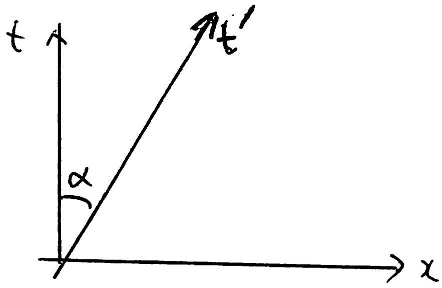
S' is an inertial observer moving with a speed v in the x direction of S.



- t' axis : t' axis is the locus of $x' = 0$, ie origin of S'

Since S' move with a speed v relative to S

So it's worldline in S should look like:



$\tan(90 - \alpha) = \frac{1}{v}$

$\Rightarrow \tan\alpha = v$

(Since $v < c \Rightarrow \alpha < 45°$)

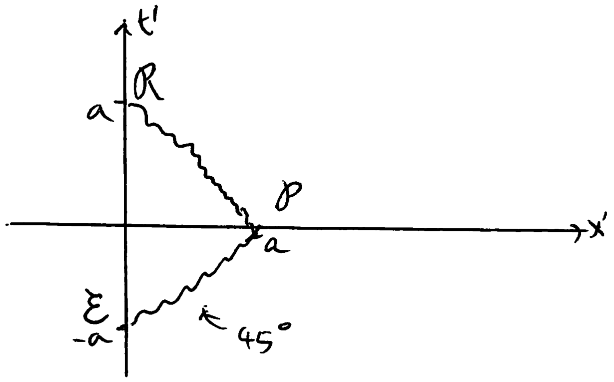
- x' axis: This is the locus of events with $t'=0$, ie those that S' measure to be simultaneous with the event $x'=t'=0$.

To identify the x' axis in the (t,x) coord system, we recall a defining property of x'-axis: Consider the process:

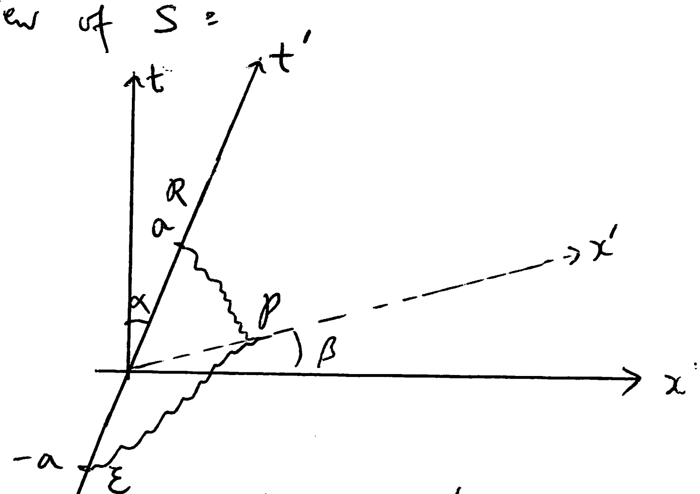
In the S' system, if we consider a light ray is emitted at $(x'=0, t'=-a)$, it will reach the x'-axis at $x'=a$ at $t'=0$.
↑ event $\mathcal{E}$ ↗ event $\mathcal{P}$

Consider a reflection there, then it will reach $x'=0$ again at $t'=a$
↗ event $\mathcal{R}$

The x'-axis can be defined as the locus of the event $\mathcal{P}$ for arb. a.



Using this physical process, we now look at it from the point of view of S:



Take an arb. point $-a$ on t', and consider the light way from $\mathcal{R}$ (backward) and $\mathcal{E}$ (forward). The intersection gives a point on the x' axis!

Ex. prove that $\beta = \alpha$.

1.6 Invariance of interval

Consider two events with a difference in coordinates given by $(\Delta t, \Delta x, \Delta y, \Delta z)$.
connected by a light beam

It is $-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 0$

We define the <u>interval</u> between any two events to be

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (1.6.1)$$

consider a EM wave emitted at the origin, At time t, the pulse arrive at a detector at (x, y, z), thus it satisfies the eqn.

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2 = 0 \qquad (1.6.2)$$

$x = \Delta x$ etc.

Similarly, in coord $\bar{S}$, we have

$$\Delta \bar{x}^2 + \Delta \bar{y}^2 + \Delta \bar{z}^2 - \Delta \bar{t}^2 = 0. \qquad (1.6.3)$$

To obtain a connection between S + $\bar{S}$ (frames), we note that the relation must be <u>linear</u> (as constructed in sec 1.5).

<u>Claim</u>: The only possible connection between the quadratic form (1.6.2) + (1.6.3) is

$$\bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{t}^2 = \lambda^2 (\Delta x^2 + \Delta y^2 + \Delta z^2 - \Delta t^2)$$

where $\lambda = \lambda(v)$ with $\lambda(0) = 1$.

<u>Pf.</u>

Linear transf. of S + $\bar{S}$: $\Delta \bar{x}^\alpha = N_{\alpha\beta} \Delta x^\beta$

$$\therefore \Delta s^2 = -(\Delta \bar{t})^2 + (\Delta \bar{x})^2 + (\Delta \bar{y})^2 + (\Delta \bar{z})^2$$

$$= \sum_{\alpha\beta=0}^{3} M_{\alpha\beta} \Delta x^\alpha \Delta x^\beta \qquad (1.6.4)$$

for some constant coefficients $M_{\alpha\beta}$

↑ indep of x^α but can depend on $\vec{v}$

Now $\Delta S=0 \iff \Delta\tilde{S}=0$

Consider $\Delta S=0$ ie. $(\Delta r)^2=(\Delta t)^2$ where $\Delta r \triangleq \sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$

Sub. into (1.6.4), we get

$$\Delta\tilde{S}^2 = M_{00}(\Delta t)^2 + 2(M_{0i}\Delta x^i)\Delta t + M_{ij}\,\Delta x^i \Delta x^j \quad (1.6.5)$$

Use Einstein summation notation:
$\sum_\alpha A_\alpha B_\alpha$ denoted as $A_\alpha B_\alpha$
ie. repeated indices are summed over and we will not write $\sum_\alpha$

This eqn. is a fun. of Δx^i:

The first term $(\Delta t)^2$ is invariant under rotations of Δx^i.

The second term (Δx^i) is a vector under " .

The third term $(\Delta x^i \Delta x^j)$ is a "second rank tensor"

learn more about this later

and has two irreducible components:

$$\Delta x^i \Delta x^j = \frac{1}{3}(\Delta x^i)^2 \delta^{ij} + \left[\Delta x^i \Delta x^j - \frac{1}{3}\delta^{ij}(\Delta x^i)^2\right]$$

$$= \underbrace{\frac{1}{3}(\Delta r)^2 \delta^{ij}}_{\text{inv. under rotations}} + \underbrace{\left[\Delta x^i \Delta x^j - \frac{1}{3}\delta^{ij}(\Delta r)^2\right]}_{\text{traceless, second rank tensor. under rotations}}$$

Since $\Delta S'=0$
this means the individual irreducible components should vanish independently

This is possible only if
$$\begin{cases} M_{0i} = 0 \\ M_{ij} = -M_{00}\delta_{ij} \end{cases} \quad (1.6.6)$$

As a result, $$\Delta\tilde{S}^2 = M_{00}\left[(\Delta t)^2 - (\Delta x^i)^2\right] = -M_{00}\,\Delta S^2 \quad (1.6.7)$$

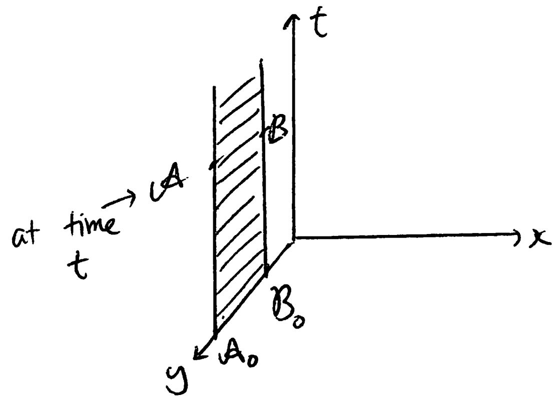
Hence $$\Delta\tilde{S}^2 = \phi(\vec{v})\,\Delta S^2 \quad (1.6.8)$$
for some function $\phi(\vec{v})$.

Next we want to show that $\phi(\vec{v}) = f(|\vec{v}|)$, i.e. it is indeed a function of $|\vec{v}|$.

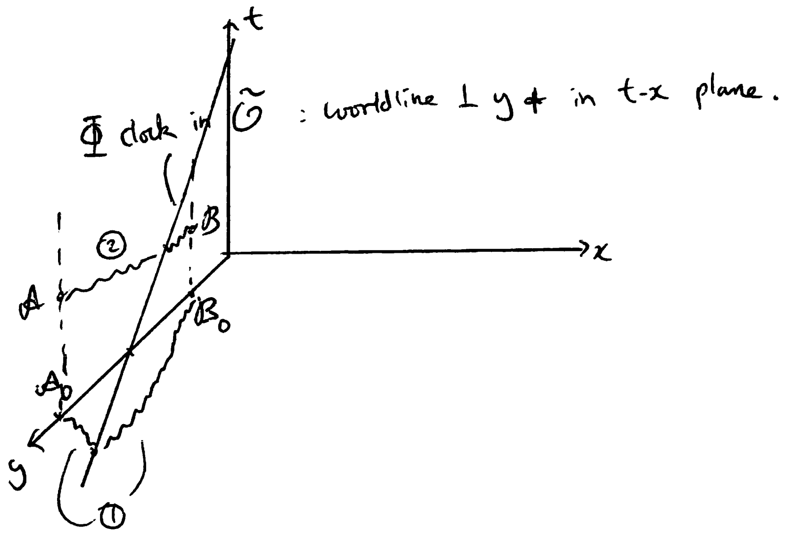
The proof is simple: 1° Consider a rod placed at the y-axis
($\perp$ to the vel. $/\!/ x$)

If we want to measure the length of the rod, we need two events A_0 & B_0 that are simultaneous at $t=0$ & occurs at the ends of the rod.

then $\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 = +(\Delta y)^2 = +(\text{length of the rod in } \mathcal{O})^2$ (1.6.9)



2° Next consider a Φ clock at rest in the frame $\tilde{\mathcal{O}}$. We claim that the events A & B are still simulataneous in $\tilde{\mathcal{O}}$. This can be seen easily by looking at the worldline of the clock:



Suppose we synchonize the ends A_0 & B_0 with the clock Φ at time $t=0$ using light (① in the figure), then it is easy to see that as the clock Φ moves, it will remain synchronized with the events A & B at the time t.

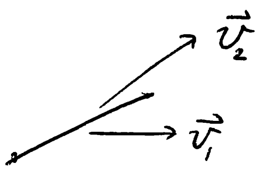
Hence the events A, B are synchronized ~~in~~ in frame $\tilde{O}$ as well.

$$\therefore (\Delta s')^2 = +(\text{length of rod in } \tilde{O})^2 \qquad (1.6.10)$$

So

$$\therefore (\text{length of rod in } \tilde{O})^2 = \phi(\vec{v})\,(\text{length of rod in } O)^2$$

However the argument remains the same so long as the velocity $\vec{v}$ is perpendicular to the rob., i.e. $\vec{v}$ can be in any direction in the x-~~z~~ plane. On the other hand, the ratio $\tilde{l}^2/l^2$ should not depend on this choice of direction as space should be isotropic.



Hence $\tilde{l}^2/l^2$ can ~~dep~~ only depend on $|\vec{v}|$.

ie $$\Delta s'^2 = \Delta s^2 \times f(|\vec{v}|) \qquad (1.6.11)$$

Finally, let us consider 3 frames, O, $\tilde{O}$, $\tilde{\tilde{O}}$ st.

$\tilde{O}$ moves with speed v in the x-direction, relative to O.
$\tilde{\tilde{O}}$ " " " " " negative x, " " $\tilde{O}$

$\therefore$ $\tilde{\tilde{O}}$ is identical to O.

Now (1.6.11) implies that:

$$\Delta \tilde{s}^2 = f(v)\,\Delta s^2$$
$$\Delta \tilde{\tilde{s}}^2 = f(v)\,\Delta \tilde{s}^2$$
and $$\Delta \tilde{\tilde{s}}^2 = \Delta s^2$$

$$\therefore f^2 = 1 \Rightarrow f = \pm 1.$$ Take $f=1$ since $\tilde{l}^2/l^2 > 0$.

Therefore, we obtain a very important result:

$$\boxed{\Delta s^2 \text{ is inv. in SR.}}$$

rmk This turns out to be true also for GR.

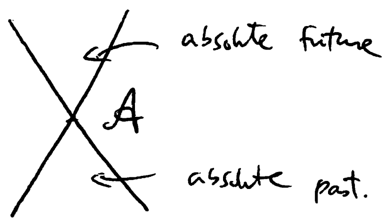
- Since Δs^2 depends on the events only and not on the observer, it can be used to classify the relation between events.

$\Delta s^2 > 0$: spacelike separated
$\Delta s^2 < 0$: timelike "
$\Delta s^2 = 0$: light like "

- The events that are lightlike separated from any event A lie on the eqn:

$$x^2+y^2+z^2-t^2=0$$

This defines a cone with A being the apex.
This is called the lightcone of A.



1-7 Lorentz transf.

Lorentz transf. is a transformation from one coordinates x^μ to another one x'^μ

So $\quad x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha \qquad \Lambda^\alpha{}_\beta,\ a^\alpha = \text{constants} \qquad (1\text{-}7\text{-}1)$

Such that $ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$ is inv. $\quad , \eta_{\alpha\beta} = (-1, +1, +1, +1)$

This implies $\quad \eta_{\gamma\delta} = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} \qquad (1\text{-}7\text{-}2)$

- The set of all Lorentz transf. (1-7-1) form a group called the inhomogeneous Lorentz group or the <u>Poincare group.</u>

The subset $a^\alpha = 0$ is called the <u>homogeneous Lorentz group</u>.

Note that (1-7-2) implies that : $(\Lambda^0{}_0)^2 \geq 1 \quad , \quad \text{Det}\,\Lambda = \pm 1 \qquad (1\text{-}7\text{-}3)$

↑ Putting $\gamma = \delta = 0$ in (1-7-2) yields

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \geq 1$$

Thus in general the homogeneous Lorentz group can be further divided into

	$\text{Det}\,\Lambda = +1$	$\text{Det}\,\Lambda = -1$
$\Lambda^0{}_0 \geq 1$	①	②
$\Lambda^0{}_0 \leq -1$	③	④

① is called the proper homogeous Lorentz group. It is connected with the identity transformation $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta$.

②, ③, ④ are called improper Lorentz transf. since it is impossible to change parameters continuously to jump from

$\Lambda^0{}_0 \leq -1$ to $\Lambda^0{}_0 \geq +1$

or $\operatorname{Det}\Lambda = -1$ to $\operatorname{Det}\Lambda = +1$.

The improper transformations ② involves a space inversion $P: x^i \to -x^i$.

The " " ④ " a time reversal $T: t \to -t$.

The " " ③ " a product of P & T.

We know now P, T or PT is not an exact symmetry of nature. Thus when we construct a fundamental theory, we usual only require the theory to be Lorentz invariant under the proper ~~homog~~ Lorentz group.

- The proper homogeneous Lorentz transformations have a further subgroup consisting of rotations:

$$\Lambda^i{}_j = R^i{}_j \;, \quad \Lambda^i{}_0 = \Lambda^0{}_i = 0 \,, \quad \Lambda^0{}_0 = 1$$

$$\text{st } |R| = 1 \;, \quad R^T R = 1$$

Together with the spacetime translations $x^\alpha \to x^\alpha + a^\alpha$, they form the <u>Galileo group</u>.

But Lorentz group also contains the <u>boost</u> transformations that change the velocity of the coordinates frame. This is defined by:

$$dx'^\alpha = \Lambda^\alpha{}_\beta \, dx^\beta$$

Suppose a particle is at rest in O and a second observer O' see it moving at velocity $\vec{v}$, then

$$d\vec{x} = 0$$

$$\text{and } \; dx'^i = \Lambda^i{}_0 \, dt$$

$$dt' = \Lambda^0{}_0 \, dt$$

$\therefore \ v^i = \frac{dx'^i}{dt'} \Rightarrow \Lambda^i{}_0 = v_i \Lambda^0{}_0$

Sub. into $-1 = (\Lambda^i{}_0)^2 - (\Lambda^0{}_0)^2$

we get $\begin{cases} \Lambda^0{}_0 = \gamma \\ \Lambda^i{}_0 = \gamma v_i \end{cases}$ $\qquad \gamma = (1-v^2)^{-1/2}$

The other components of $\Lambda^\alpha{}_\beta$ are not uniquely determined because if $\Lambda^\alpha{}_\beta$ carries a particle from rest to velocity $\vec{v}$, then $\Lambda^\alpha{}_\gamma R^\gamma{}_\beta$ (with R an arb. rotation) also does that.

A convenient choice that satisfies (1-7-2) is given by

$$\begin{cases} \Lambda^i{}_j = \delta_{ij} + v_i v_j \dfrac{\gamma - 1}{v^2} \\ \Lambda^0{}_j = \gamma v_j \end{cases} \qquad (1\text{-}7\text{-}4)$$

18 Some simple consequences of Lorentz transf.

Time dilation: Suppose an observer at rest see the two ticks of a clock (at rest to him) separated by a spacetime interval $d\vec{x} = 0$, $dt = \Delta t$. Hence $ds^2 = -(\Delta t)^2$

To another observer moving with a speed $\vec{v}$, he will observe that the two ticks are separated by a time interval dt' and a space interval $d\vec{x}' = \vec{v}\, dt'$.

Therefore $ds'^2 = -\sqrt{1-v^2}\, dt'$

$$\therefore \quad dt' = \Delta t\, \gamma \quad , \quad \gamma = \frac{1}{\sqrt{1-v^2}} \geq 1 \qquad (1.8.1)$$

This is called the time dilation of moving clock.

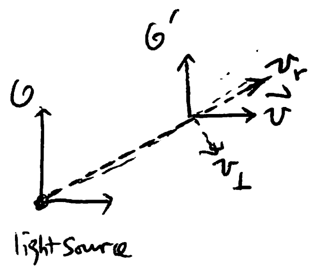
This has been verified experimentally. e.g. a moving particle has a longer lifetime than it has at rest by a factor of γ.

Doppler effect: If the clock is a moving source of light of freq $\nu = 1/\Delta t$, then there is an additional effect called Doppler effect:

The time between <u>emission</u> of successive wavefront is given by

$$dt' = \Delta t\, \gamma$$

However during this time the distance from the observer to the light source will have increased by an amount $v_r\, dt'$ where v_r is the component of $\vec{v}$ along the direction from observer to the light source.



Hence the period between reception of wavefront will be

$$dt_0 = \left(1 + \frac{v_r}{c}\right) dt' = \frac{(1 + v_r)}{\sqrt{1-v^2}}\, \Delta t$$

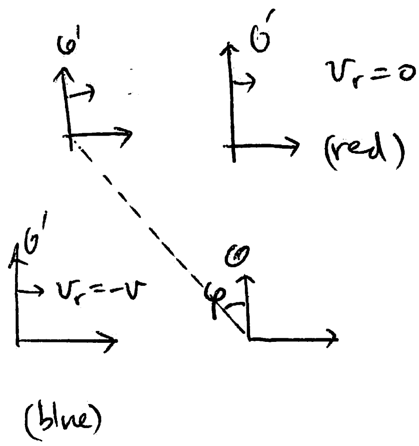
$$\therefore \frac{\nu_{obs}}{\nu} = \frac{\sqrt{1-v^2}}{1+v_r} \qquad (1.82)$$

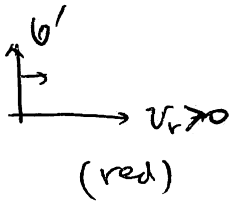
$\therefore$ if $v_r > 0$, the $\frac{\nu_{obs}}{\nu} < 1$ ie a redshift

if $v_r = 0$, " a pure time dilation red shift

if $v_r = -v$, then $\frac{\nu_{obs}}{\nu} = \sqrt{\frac{1+v}{1-v}} > 1$, ie a blue shift.

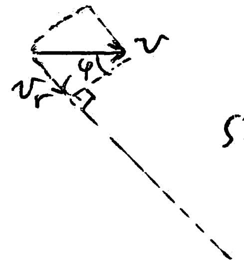
The transition from blue to red shift occurs at a certain angle φ :





such that

$$v_r = \sqrt{1-v^2} - 1$$



$$\sin\varphi = \frac{\sqrt{1-v^2}-1}{v}$$