11. Schwarzschild geom & BH

11-1 Classical Test of GR

general EoM

Consider metric of the form (std form)

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (11.1.1)$$

geodesic eqn: $\dfrac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda}\dfrac{dx^\nu}{dp}\dfrac{dx^\lambda}{dp} = 0$ (can take $p = \tau$ for massive particle)

We get $$0 = \frac{d^2r}{dp^2} + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\varphi}{dp}\right)^2 + \frac{B'}{2A}\left(\frac{dt}{dp}\right)^2 \qquad (11.1.2)$$

$$0 = \frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - \sin\theta\cos\theta\left(\frac{d\varphi}{dp}\right)^2 \qquad (11.1.3)$$

$$0 = \frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{d\varphi}{dp}\frac{dr}{dp} + 2\cot\theta\frac{d\varphi}{dp}\frac{d\theta}{dp} \qquad (11.1.4)$$

$$0 = \frac{d^2t}{dp^2} + \frac{B'}{B}\frac{dt}{dp}\frac{dr}{dp} \qquad (11.1.5)$$

metric is isotropic ~~⇒ which can~~

⇒ orbit must be on a plane

WLOG, can take it to be on equator $\theta = \pi/2$

(11.1.3) is satisfied.

Divide (11.1.4) by $\dfrac{d\varphi}{dp}$ ⇒ $\dfrac{d}{dp}\left(\ln\dfrac{d\varphi}{dp} + \ln r^2\right) = 0 \qquad (11.1.6)$

(11.1.5) by $\dfrac{dt}{dp}$ ⇒ $\dfrac{d}{dp}\left(\ln\dfrac{dt}{dp} + \ln B\right) = 0 \qquad (11.1.7)$

can solve (11.1.7) and get

$$\frac{dt}{dp} = \frac{1}{B(r)} \qquad (11.1.8)$$

with one integration constant absorbed into the def. of p.

(11.1.6) $\Rightarrow r^2 \frac{d\varphi}{dp} = J$ = const. of motion (11.1.9)

then (11.1.2) gives $\frac{d^2r}{dp^2} + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{r^3 A} + \frac{B'}{2AB^2}$

multiply by $2A\frac{dr}{dp}$ $\Rightarrow$ $\frac{d}{dp}\left[A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B}\right] = 0$

$$\Rightarrow A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E \quad = \text{const.} \qquad (11.1.10)$$

Compared with (11.1.1),
This implies $d\tau^2 = E\, dp^2$ (11.1.11)

$\therefore$ for massive particle, take $E > 0$

γ , take $E = 0$

Now since $A(r) > 0$, $\therefore$ $\frac{J^2}{r^2} + E \leq \frac{1}{B(r)}$ (11.1.12)

$\uparrow$

equality at max. r.

Eliminate p,

$$\begin{cases} r^2 \dfrac{d\varphi}{dt} = J\, B(r) \\ \dfrac{A}{B^2}\left(\dfrac{dr}{dt}\right)^2 + \dfrac{J^2}{r^2} - \dfrac{1}{B(r)} = -E \\ d\tau^2 = E B^2 dt^2 \end{cases} \qquad \text{— (11.1.13)}$$

Weak field approx: J^2/r^2, $\left(\frac{dr}{dt}\right)^2$, $A-1$, $B-1 \simeq 2\phi$ all small and kept to first order.

then (11.1.13) become

$$\begin{cases} r^2 \dfrac{d\varphi}{dt} \simeq J \\ \dfrac{1}{2}\left(\dfrac{dr}{dt}\right)^2 + \dfrac{J^2}{2r^2} + \phi \simeq \dfrac{1-E}{2} \end{cases} \qquad (11.1.14)$$

Same eqn as in Newtonian theory, with $\frac{1-E}{2}$ being the energy per unit mass.

- To solve (11.1.14), consider first the case of a closed circular orbit of radius R.

$$\frac{dr}{dt} = 0 \Rightarrow \frac{J^2}{R^2} - \frac{1}{B(R)} + E = 0$$

In Equilibrium $\Rightarrow$ $$-\frac{2J^2}{R^3} + \frac{B'}{B^2} = 0$$

eliminate J,

$$\begin{cases} E = \dfrac{1}{B(R)}\left(1 - \dfrac{R\, B'(R)}{2B(R)}\right) \\ J^2 = \dfrac{B'(R)\, R^3}{2B^2(R)} \end{cases} \qquad (11.1.15)$$

$$\therefore \begin{cases} \dfrac{d\varphi}{dt} = \dfrac{\cancel{B(R)}}{R^2} J = \sqrt{\dfrac{B'(R)}{2R}} \\ \\ \dfrac{d\tau}{dt} = \sqrt{E}\, B(R) = \sqrt{B(R) - \frac{1}{2} R\, B'(R)} \end{cases} \qquad (11.1.16)$$

for Schwarzschild soln.

For $B(r) = 1 - \dfrac{2M}{r} + \underbrace{2(\beta-\gamma)\dfrac{M^2}{r^2}}_{} + \cdots$

zero for Schwarzschild soln. $\because \beta = \gamma$

But historically used to parametrize soln

to alternative of Einstein relativity

$$\text{get} \begin{cases} \dfrac{d\varphi}{dt} = \sqrt{\dfrac{M}{R^3}} \left(1 - \dfrac{(\beta-\gamma)M}{R} + \cdots\right) \\ \\ \dfrac{d\tau}{dt} = 1 - \dfrac{3M}{R} + \cdots \end{cases} \qquad (11.1.17)$$

we are interested in r as a fn. of φ ~~most~~ more than r as a fn of t.

(Shape of orbit)

Eliminate dp from (11.1.10) & (11.1.9),

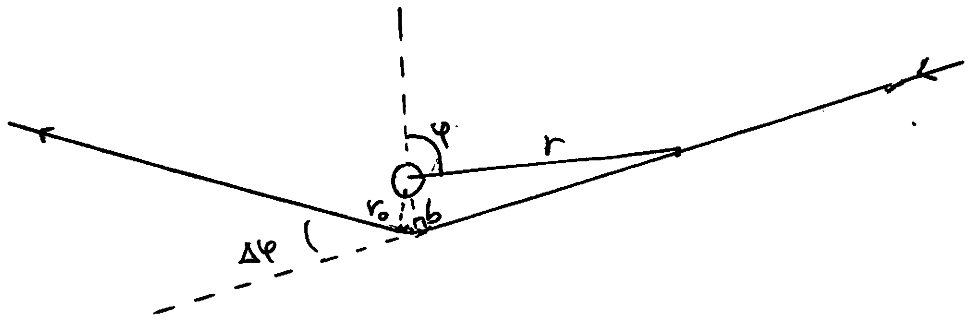
$$\frac{A(r)}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2}$$

$$\Rightarrow \varphi = \pm \int \frac{A^{\frac{1}{2}}(r)\, dr}{r^2\left(\dfrac{1}{J^2 B(r)} - \dfrac{E}{J^2} - \dfrac{1}{r^2}\right)^{\frac{1}{2}}} \qquad (11.1.18)$$

Classical Tests of GR

1° Unbounded orbits: Deflection of light by Sun

Consider a particle or photon approaching the sun from a great distances.



At infinity, the metric becomes Minkowskian, $A(\infty) = B(\infty) = 1$

and particles move on a straight line with velocity V.

$$\Rightarrow \begin{cases} b \simeq r\sin(\varphi - \varphi_\infty) \simeq r(\varphi - \varphi_\infty) \\ -V = \frac{d}{dt}\left(r\cos(\varphi - \varphi_\infty)\right) \simeq \frac{dr}{dt} \end{cases} \qquad (11\text{-}1\text{-}19)$$

Constant of motion: $\begin{cases} J B(r) = r^2 \frac{d\varphi}{dt} \\ -E = \frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} \end{cases}$

$$\Rightarrow \begin{cases} J = bV \\ -E = V^2 - 1 \end{cases}$$

$$(b, V) \to (E, J) \quad \text{ie} \begin{cases} J = b^2 V \\ E = 1 - V^2 \end{cases} \qquad (11\text{-}1\text{-}20)$$

Convenient to express J in terms of the distance r_0 of closest approach to the sun, rather than b.

At r_0, $\frac{dr}{d\varphi} = 0 \Rightarrow \quad V^2 - 1 = \frac{J^2}{r_0^2} - \frac{1}{B(r_0)} \Rightarrow \quad J = r_0\left(\frac{1}{B(r_0)} + V^2 - 1\right)^{1/2}.$

The orbit $\varphi = \int \frac{A^{1/2} dr}{r^2 \left(\frac{1}{J^2 B} - \frac{E}{J^2} - \frac{1}{r^2} \right)^{1/2}}$

$$= \varphi_\infty + \int_r^\infty \frac{A^{1/2}\, dr}{r^2 \left(\frac{1}{r_0^2} \left[\frac{1}{B(r)} - 1 + v^2 \right] \left[\frac{1}{B(r_0)} - 1 + v^2 \right]^{-1} - \frac{1}{r^2} \right)^{1/2}} \qquad (11.1.21)$$

deflection of orbit $= \Delta\varphi = 2\,|\varphi(r_0) - \varphi_\infty| - \pi$

$\Delta\varphi > 0$: bent toward the sun

$\Delta\varphi < 0$: bent away from the sun

For photon, $V^2 = 1 \Rightarrow$ $$\varphi(r) - \varphi_\infty = \int_r^\infty \frac{A^{1/2}}{r} \frac{1}{\sqrt{\left(\frac{r}{r_0}\right)^2 \frac{B(r_0)}{B(r)} - 1}}\, dr \qquad (11.1.22)$$

~~Expand~~ Can evaluate it in terms of elliptic integral.

Instead let's expand integrand in terms of small MG/r, MG/r_0 and then integrate.

the result is

$$\varphi(r) - \varphi_\infty = \int_r^\infty \frac{dr}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} \left(1 + \frac{MG}{r} + \frac{MGr}{r_0(r+r_0)} + \cdots \right)$$

$$= \sin^{-1}\left(\frac{r_0}{r}\right) + \frac{MG}{r_0} \left(2 - \sqrt{1 - \left(\frac{r_0}{r}\right)^2} - \sqrt{\frac{r-r_0}{r+r_0}} + \cdots \right) \qquad (11.1.23)$$

To first order in MG/r_0, $\Delta\varphi = \frac{4MG}{r_0}$ $\qquad (11.1.24)$

For sun, $M_\odot = 1.97 \times 10^{33}$ g, $R_\odot = 6.95 \times 10^5$ km, $\Delta\varphi = \frac{R_\odot}{r_0}\theta_\odot$, $\theta_\odot = \frac{4M_\odot G}{R_\odot} = 1.75''$.

2° Bound orbits : Precession of Perihelia

Consider a test particle bound in an orbit around the sun.

At perihelia and aphelia, r reaches its min ~~max~~ & max values r_- & r_+,

at which points $\frac{dr}{d\varphi} = 0$. So

$$\frac{1}{r_\pm^2} - \frac{1}{J^2 B(r_\pm)} = \frac{-E}{J^2} \qquad (11.1.25)$$

From these two eqn. we can get:

$$\begin{cases} E = \dfrac{r_+^2/B_+ - r_-^2/B_-}{r_+^2 - r_-^2} \\[2ex] J^2 = \dfrac{1/B_+ - 1/B_-}{1/r_+^2 - 1/r_-^2} \end{cases} \qquad (11.1.26)$$

The angle swept out by the position vector as r increases from r_- is

given by $$\varphi(r) = \varphi(r_-) + \int_{r_-}^{r} \sqrt{A(r)} \left(\frac{1}{J^2 B(r)} - \frac{E}{J^2} - \frac{1}{r^2} \right)^{-1/2} \frac{dr}{r^2}$$

or using E^2, J^2 above,

$$\varphi(r) = \varphi(r_-) = \int_{r_-}^{r} dr \frac{\sqrt{A(r)}}{r^2} \left[\frac{r_-^2 (B^{-1} - B_-^{-1}) - r_+^2 (B^{-1} - B_+^{-1})}{r_+^2 r_-^2 (B_+^{-1} - B_-^{-1})} - \frac{1}{r^2} \right]^{-1/2} \qquad (11.1.27)$$

the change in φ as r decrease from r_+ to r_- is the same as the change as r increase from r_- to r_+,

so total change in φ per revolution is

$$2(\varphi(r_+) - \varphi(r_-))$$

This would be a close ellipse if this is 2π. So in general the orbit processes in each revolution by an angle

$$\Delta\varphi = 2\left(\varphi(r_+) - \varphi(r_-)\right) - 2\pi, \qquad (11.1.28)$$

Using the exact formula of $A(r)$, $B(r)$ in the Schwarzschild soln:

$$A(r)^{-1} = B(r) = 1 - \frac{2MG}{r} \qquad (11.1.29)$$

we yield formula for $\Delta\varphi$ in terms of elliptic integral.

To evaluate it we will then need to expand in MG/r, $MG/r_\pm$.

Instead one can expand first + integrate, the result is

$$\Delta\varphi = \frac{6\pi MG}{L}. \qquad \text{radius/revolution} \qquad (11.1.30)$$

where L is defined by $\frac{1}{L} \doteq \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \qquad (11.1.31)$

called the semilatus rectum.

we can also introduce eccentricity e and the semimajor axis a,

$$r_\pm = (1 \pm e)\,a$$

Hence we can determine L from a + e:

$$L = (1 - e^2)\,a$$

The fact that $\Delta\varphi > 0$ means that the whole orbit should precess in the same direction as the motion of the test particle.

3° Radar Echo Delay

The classic tests discussed above deal with the shape of the trajectories of photon & planets.

Another test ~~...~~ using Radar echo measurable depending on the "time" only.

Let us calculate the time required for a radar signal to go from

$r = r_1, \ \theta = \pi/2, \ \varphi = \varphi_1$

to $r = r_2, \ \theta = \pi/2, \ \varphi = \varphi_2$.

Time history of orbit : $$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = -\mathcal{E} = 0 \quad (11.1\text{-}32)$$ for light ray.

~~for li~~

At $r = r_0$ (closest ~~app~~ approach), $\frac{dr}{dt} = 0$

$$\Rightarrow J^2 = r_0^2 / B(r_0) \quad (11.1\text{-}33)$$

$$\Rightarrow \frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \left(\frac{r_0}{r}\right)^2 B^{-1}(r_0) - B^{-1}(r) = 0$$

$$\Rightarrow t(r, r_0) = \int_{r_0}^{r} \sqrt{\frac{A(r)/B(r)}{1 - \frac{B(r)}{B(r_0)}\left(\frac{r_0}{r}\right)^2}}\, dr \quad (11.1\text{-}34)$$

Total time of the echo $= t_{12} = t(r_1, r_0) + t(r_2, r_0)$ (11.1-35)

~~We~~ Using the small $\frac{MG}{r_1}$, $\frac{MG}{r_0}$ expansion, we obtain

$$t(r, r_0) \simeq \int_{r_0}^{r}\left(1 - \frac{r_0^2}{r^2}\right)^{-1/2}\left[1 + \frac{2MG}{r} + \frac{MG r_0}{r(r+r_0)}\right] dr \quad (11.1.36)$$

Elementary integral and get

$$t(r,r_0) = \sqrt{r^2-r_0^2} + 2MG \ln\left(\frac{r+\sqrt{r^2-r_0^2}}{r_0}\right) + MG\sqrt{\frac{r-r_0}{r+r_0}} \qquad (11.1\text{-}37)$$

SR expected result if light travelled in str. line

delay due to GR effect.

Confirmed by experiment of radar signals goes ~~for~~ to Mercury & return.

11-2 Horizon of Schwarzschild Blackhole

Buchdahl limit says a stable star has size $R \geq \frac{9}{8} R_S$, so the surface R_S is always in the interior of star and analysis of the nature of singularity at $r=R_S$ & $r=0$ of the Schwarzschild soln. is irrelevant. On the other hand, for a star under grav. collapse, the region $r \leq 2M$ is very relevant to the description of the endpoint of this collapse.

Coordinate singularities

There is a difference between geometric singularity and coord. singularity.

For example, $\mathbb{R}^2$ has metric $ds^2 = dx^2 + dy^2$. No singularity anywhere.
But in spherical coords $ds^2 = dr^2 + r^2 d\theta^2$. At the origin $r=0$, θ can be anything, yet corresponds to the same point.

$r=0$ is a coordinate singularity.

Horizon

Consider the Schwarzschild metric,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (11\text{-}2\text{-}1)$$

$r=2M$ is a singularity. Is it real or not?

one way to answer this question is to send in a particle to probe the geometry.

Recall that since the metric is indep. of t & ϕ, so p_0 & p_ϕ are constant of motion.

$$\left(\because\ m\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\mu\nu,\beta}\, p^\mu p^\nu\right)$$

Let us call
$$\begin{cases} \tilde{E} = -p_0/m & \text{Energy} \\ \tilde{L} = p_\phi/m & \text{Angular mom. per unit mass} \end{cases} \qquad (11\text{-}2\text{-}2)$$

The EOM for the particle can be obtained from the conservation law (as usual)

$$p_\mu p^\mu = -m^2 \qquad (11\text{-}2\text{-}3)$$

Now,

$$\begin{cases} p^0 = g^{00} p_0 = m\left(1-\frac{2M}{r}\right)^{-1}\tilde{E} \\ p^r = m\frac{dr}{d\tau} \\ p^\phi = g^{\phi\phi} p_\phi = \frac{m}{r^2}\tilde{L} \\ p^\theta = 0 \qquad \text{since } \theta = \text{constant as motion is planar.} \end{cases}$$

$$\Rightarrow \cancel{-m^2} = \cancel{-m^2}\left(1-\frac{2M}{r}\right)^{-1}\tilde{E}^2 + \cancel{m^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \cancel{m^2}\frac{\tilde{L}^2}{r^2}$$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) \qquad (11\text{-}2\text{-}4)$$

Now consider a particle falling in to the Schwarzschild metric in the radial direction,

then $\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - 1 + \frac{2M}{r}$

minus sign for infalling

$$\text{or } d\tau = -\frac{dr}{\left(\tilde{E}-1+\frac{2M}{r}\right)^{1/2}} \qquad (11\text{-}2\text{-}5)$$

What is the proper time it takes for it to go from a finite distance R to the surface $r=2M$?

For $\tilde{E}>1$, the integral is obviously finite.

For $\tilde{E}=1$, $\Delta\tau = \frac{4M}{3}\left(\frac{r}{2M}\right)^{3/2}\Big|_{2M}^{R}$ = finite

For $\tilde{E}<1$, in order for integrand to be well defined, need $\frac{2M}{r} > 1-\tilde{E}$

$$r < \frac{2M}{1-\tilde{E}}$$

This is the only constraint, so long as the initial position $R < \frac{2M}{1-\tilde{E}}$ then integral is again finite.

So the particle can always reach the horizon at a finite proper time!

↳ $r = 2M$ is a coord. singularity!

What about coordinate time?

$$\frac{dt}{d\tau} = U^0 = g^{00}\frac{p_0}{m} = -\tilde{E}g^{00} = \tilde{E}\left(1-\frac{2M}{r}\right)^{-1}$$

$$\Rightarrow dt = \frac{\tilde{E}\,d\tau}{1-2M/r} = \frac{\tilde{E}\,dr}{\left(1-\frac{2M}{r}\right)\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{1/2}} \qquad (11\text{-}2\text{-}6)$$

For simplicity consider the case $\tilde{E}=1$, and near $r=2M$,
let $r = 2M+\varepsilon$, then $dt = \dfrac{-(\varepsilon+2M)^{3/2}\,d\varepsilon}{(2M)^{1/2}\,\varepsilon}$

Consider the region near $r=2M$, i.e. ε small, then

$$\Delta t \sim \ln\varepsilon \quad \text{for small } \varepsilon \qquad (11\text{-}2\text{-}7)$$

diverges

Therefore it takes an infinite amount of coord time to reach the surface $r=2M$!

physically, this is due to ∞ red shift caused by the grav. field near $r=2M$.

What happen after passing through the horizon?
- Next consider inside the horizon.

For $r<2M$, $\quad 1-\frac{2M}{r} < 0$ and

$$ds^2 = \underbrace{-\left(1-\frac{2M}{r}\right)}_{>0}dt^2 + \underbrace{\left(1-\frac{2M}{r}\right)^{-1}}_{<0}dr^2 + r^2 d\Omega^2 \qquad (11\text{-}2\text{-}8)$$

r becomes time like and t becomes spacelike!

As trajectory must follow a time like curve, ~~t~~ r must keep increasing

and so must reach $r=0$!

it is a curvature singularity!

Q. Can one send out a signal (eg. photon) to feel the outside world about the interior region?
As the photon must follow a geodesic & go forward in time, it means r must decreases.

Hence cannot reach $r=2M$!

Everything inside is trapped, and doomed to reach the singularity at $r=0$.

The surface $r=2M$ is called a <u>horizon</u>.
A spacetime where the metric hold all the way down to $r=0$ is called a BH.

- <u>Maximal analytic continuation of the Schwarzschild soln.</u>

The coord. system (11-2-1) has the properties that at $r\to\infty$,

$$ds^2 \to (ds^2)_{flat}.$$

ie asymptotically flat.

This coord is good for an observer very far away from the BH.

However the surface $r=2M$ appears to be singular in this coord. system.

It would be nice to employ a different coord system such that the singularity at $r=2M$ does not appear.

<u>Kruskal-Szekeres coord.</u>

Defined
$$\begin{cases} u = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M} \cosh(t/4M) \\ v = \quad " \quad " \quad \sinh(t/4M) \end{cases} \quad \text{for } r>2M \qquad (11\text{-}2\text{-}9a)$$

$$\begin{cases} u = \left(1-\frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh(t/4M) \\ v = \quad " \quad " \quad \cosh(t/4M) \end{cases} \quad \text{for } r<2M \qquad (11\text{-}2\text{-}9b)$$

Then $ds^2 = -\frac{32M^3}{r} e^{-r/2M} (dv^2 - du^2) + r^2 d\Omega^2$ (11-2-10)

$\begin{pmatrix} v \sim T \\ u \sim R \end{pmatrix}$

where $r = r(u,v)$ is determined by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = u^2 - v^2 \qquad (11\text{-}2\text{-}11)$$

Note:

1° The metric is regular at $r=2M$! It covers the entire spacetime and is well behaved everywhere.

2° The radial null line is given by $du = \pm dv$

The light cone is always $45°$.. Light speed is $c=1$ always.

3° The Schwarzschild horizon is at .

$$u = \pm v$$

ie it is a null surface

4.° The surface of constant r is given by

$$u^2 - v^2 = \text{constant} \qquad (11\text{-}2\text{-}12a)$$

The surface of constant t is given by

$$\frac{v}{u} = \tanh\left(t/4M\right) \qquad (11\text{-}2\text{-}12b)$$

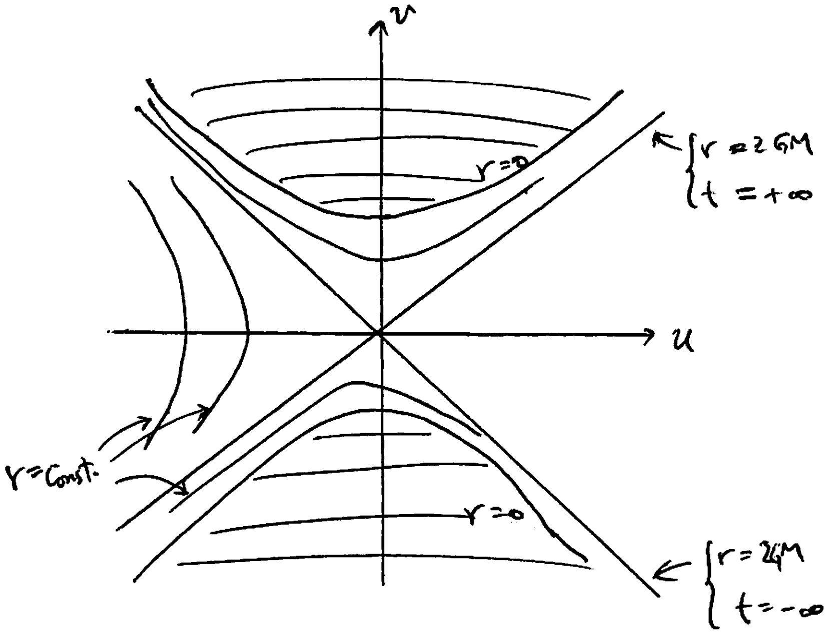
5.° The coord (v, u) should be allowed to range over every value they can take without hitting the singularity at $r=0$ (ie. still a soln to Einstein eqn).

The allowed region is therefore :

$$-\infty < u < \infty \qquad (11\text{-}2\text{-}13)$$

$$v^2 < u^2 + 1$$

see (11-2-11)

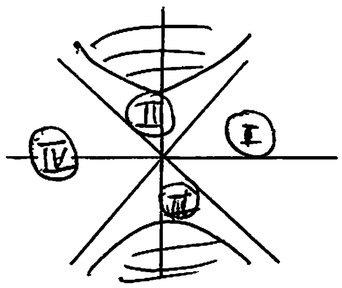


Shaded region are not allowed by (11-2-9)

Kruskal diagram

The original Schwarzschild Coord. (t, r) were good for $r > 2M$, which is only a part of the manifold described in the Kruskal diagram.

Let us divide the Kruskal diagram into 4 regions:



I : $r > 2M$ is the original Schwarzschild soln.

II : This region is reached by following the future directed null ray of region I. Also any light signals sent from II will remain in II and eventually hit the singularity. We call it a BH.

III : " " " past directed null ray of region I. Any observer in III must have originated in the singularity and within a finite time, must leave region III. It has exactly the time reversed properties of BH and is called a white Hole.

IV : This region has properties identical to original region I. It represents another asymptotically flat spacetime which lies inside the radius $r = 2M$.

Note that IV cannot be reached from our region I either forward or backward in time

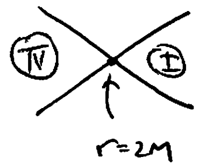
those rays will always end up on the singularity

Nor can anybody from over there reach us.

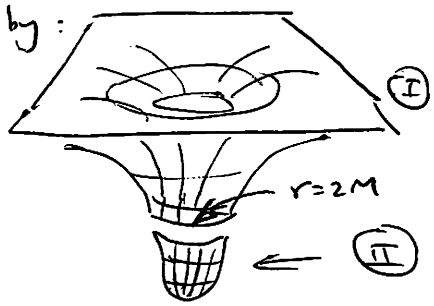
To understand better the geometry, let's look at ~~the~~ the geometry of the slice at $v=0$.

We have the region I & IV being connected at the "point" $r=2M$

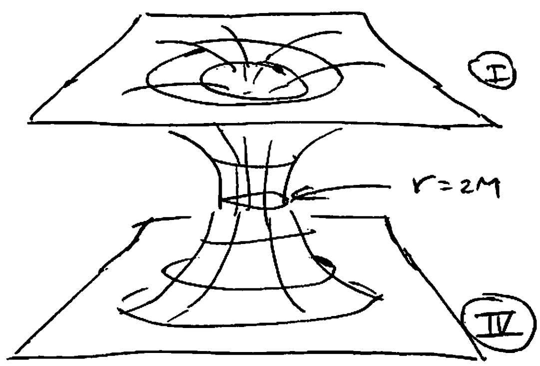
this is actually a surface S^2 of radius $r=2M$



Now I is represented by:



∴ The slice can be represented as



We can think of this as having region I connected to region IV by a <u>Wormhole</u> (or Einstein-Rosen bridge), a neck like region joining two distinct regions, which are asymptotically flat.

Movie : star trek

A classification of stars & its history of life

The Kruskal spacetime is a soln to the Einstein eqn., it is a remarkable soln. including not only the BH, but also a white hole and an additional asymptotically flat region, connected to our universe by a wormhole.

However, it is a highly idealized soln.
- completely spherically symmetric
- completely free of energy momentum throughout spacetime

the existence of matter will likely alter the picture!

In the presence of matter, we have to take into account of the interior soln.

We have the Buchdahl's theorem which says that any static, spherically symm. soln. must have a total mass $M < \frac{4R}{9G}$ or $R > \frac{9}{8} R_s$

astrophysical objects (static, sph. symm. soln)
- (static) Planets : supported by material pressures. $p = p(\rho)$ ← static, essentially $T \simeq 0$ and $R \gg R_s$
- massive stars (quasi-static) : supported by a pressure which could change slowly over time, ie. $p = p(\rho(t))$ resulting in stages of grav. collapse

Massive stars : pressure comes from heat produced by fussion of nuclei into heavier ones
[main seq. stars]
ie fussion pressure P_f

↓ Fussion used up

grav. collapse

↓

white dwarf : Collapse halted by Fermi degeneracy pressure of electrons

[Binary star]

↓ if star is sufficiently massive such that $M >$ Chandrasekhar limit $\simeq \frac{M_{pe}^3}{m_H^2} \simeq 1.4 M_\odot$ then P_e is not enough to resist the grav. pull.

ie. Fermi degeneracy pressure P_e

(m_H = mass of hyrogen atom)

neutron star : electron combined with proton to form neutron

$$e + p \rightarrow n + \nu_e$$

[pulsar]

and neutrino simply fly away.

ie neutron pressure P_n

↓ if it is even more massive, exceeding the oppenheimer - Volkoff limit, $\simeq 3-4 M_\odot$ it is believed the collapse can't be stopped as we know of no other form of material that can be more densed
(research : $n \rightarrow$ quarks and form quark star)

quark star (?)
[candidate of dark matter (?)]

↓

Black hole.
[Strong X ray sources]
- Solar mass BH : endpoint of star
- Supermassive BH : History of formation not clear?