21.2.1

2.1 Definition of vector

- Consider the Minkowski space $(\mathbb{R}^4, \eta)$. Under a Lorentz transf, the coord transf as $x^\alpha \to \Lambda^\alpha{}_\beta x^\beta$

<u>Def</u>: A Lorentz vector (or 4-vector) is any quantity that undergoes the same transformation

$$V^\alpha \to V'^\alpha = \Lambda^\alpha{}_\beta V^\beta \qquad (2.1.1)$$

More precisely, this is called a <u>contravariant four-vector</u>.

<u>Def</u>: A <u>covariant 4-vector</u> U_α transforms as

$$U_\alpha \to U'_\alpha = \Lambda_\alpha{}^\beta U_\beta \qquad (2.1.2)$$

where $\Lambda_\alpha{}^\beta \triangleq \eta_{\alpha\gamma}\eta^{\beta\delta}\Lambda^\gamma{}_\delta$. Note that the indices γ & δ on the RHS are raised & lowered by η's.

and $\eta^{\beta\delta}$ is numerically the same as $\eta_{\beta\delta}$.

- Note that $\eta^{\beta\delta}\eta_{\delta\alpha} = \delta^\beta{}_\alpha = \begin{cases} 1 & \alpha=\beta \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ $\eta^{\beta\delta}$ is the inverse of $\eta_{\beta\delta}$.

It can be seen that $\Lambda_\alpha{}^\beta$ is the inverse of $\Lambda^\beta{}_\gamma$ since

$$\begin{aligned} \Lambda_\alpha{}^\beta \Lambda^\alpha{}_\gamma &= \eta_{\alpha\kappa}\eta^{\beta\delta}\Lambda^\kappa{}_\delta \Lambda^\alpha{}_\gamma \\ &= \eta^{\beta\delta}\eta_{\delta\gamma} \\ &= \delta^\beta{}_\gamma \end{aligned}$$

ie $\Lambda_\alpha{}^\beta = (\Lambda^{-1})^\beta{}_\alpha \qquad (2.1.3)$

• **Lemma**: The scalar product of a contravariant with a covariant 4-vector is invariant:

$$U'_\alpha V'^\alpha = \underbrace{\Lambda_\alpha{}^\gamma \Lambda^\alpha{}_\beta}_{\delta^\gamma{}_\beta} U_\gamma V^\beta = U_\beta V^\beta$$

\#

• **Def**: raise & lower of indices

We use η to raise & lower the indices

e.g. given a contravariant 4-vector V^α, we can lower its indices and define

$$V_\alpha \triangleq \eta_{\alpha\delta} V^\beta \qquad (2.1\text{-}4)$$

Similarly, given U_α, we can define

$$U^\alpha \triangleq \eta^{\alpha\beta} U_\beta \qquad (2.1\text{-}5)$$

Lemma: (2.1.4) indeed gives a covariant 4-vector.
(2-1-5) " " a contravariant 4-vector.

Pf. $V'_\alpha = \eta_{\alpha\beta} V'^\beta = \eta_{\alpha\beta} \Lambda^\beta{}_\gamma V^\gamma = \underbrace{\eta_{\alpha\beta} \Lambda^\beta{}_\gamma \eta^{\gamma\delta}}_{\Lambda_\alpha{}^\delta} V_\delta$ ✓

similar for (2.1.5) \#

Examples of 4 vector: dx^α is a contravariant 4-vector

$\frac{\partial}{\partial x^\alpha}$ is a covariant 4-vector

pf:

$$\frac{\partial}{\partial x^\alpha} \to \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} = \Lambda_\alpha{}^\beta \frac{\partial}{\partial x^\beta} \qquad \left(\begin{array}{c} x'^\alpha = \Lambda^\alpha{}_\gamma x^\gamma \\ \Leftrightarrow x^\gamma = x'^\alpha \Lambda_\alpha{}^\gamma \end{array} \right)$$

//

Natural to denote $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$

$\Box^2 \triangleq \eta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$ is called the ~~de~~ d'Alembertian operator.

$$= \nabla^2 - \frac{\partial^2}{\partial t^2}$$

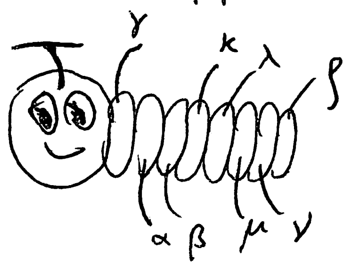
Generalization of the Laplace operator

__Tensors__: More generally, we could consider quantities with more complicated transformation properties, e.g.

$$T^\gamma{}_{\alpha\beta} \to T'^\gamma{}_{\alpha\beta} = \Lambda^\gamma{}_\delta \Lambda_\alpha{}^\varepsilon \Lambda_\beta{}^\kappa T^\delta{}_{\varepsilon\kappa}$$

Note that it is important to keep track of the position of the indices. One can do this easily by remembering that on each column there can only be one index, either up or down.

__NB.__ A tensor is like a caterpillar with lots of legs:



Scalar: A tensor without any indices is called a scalar. It is invariant under Lorentz transformation:

$$\phi \to \phi' = \phi$$

- The above is the <u>algebraic way</u> to define a tensor. Let us consider a more <u>geometrical</u> approach in the following:

Def. A (Contravariant) vector is a geometrical object which transforms in the same way as a displacement vector $\Delta\vec{x}$.

$\leftarrow$ used as a ref. for defining vector.

It is convenient to use a coordinate system. With respect to a coord. system, we can read off the components of the vector, we denote this process as:

$$\Delta\vec{x} \xrightarrow[\mathcal{O}]{} (\Delta t, \Delta x, \Delta y, \Delta z) \triangleq \{\Delta x^\alpha\}$$

$\leftarrow$ convenient notation

$$\vec{A} \xrightarrow[\mathcal{O}]{} (A^0, A^1, A^2, A^3) \triangleq \{A^\alpha\}$$

With respect to a different frame $\mathcal{O}'$, the same vector has different components:

$$\Delta\vec{x} \xrightarrow[\mathcal{O}']{} (\Delta t', \Delta x', \Delta y', \Delta z') \triangleq \{\Delta x'^\alpha\}$$

$$\vec{A} \xrightarrow[\mathcal{O}']{} (A'^0, A'^1, A'^2, A'^3) \triangleq \{A'^\alpha\}$$

Def. A covariant vector is an object which transform in the same way as $\vec{\partial}$.

- Note that addition & scalar multiple of vectors still give a vector:

$$\vec{A} + \vec{B}, \quad \mu\vec{A} \quad \text{are vectors if } \vec{A}, \vec{B} \text{ are.}$$

- Similarly, one can define tensor of type $\binom{m}{n}$ as an object which transforms in the same way as a (direct) product of m contravariant vectors & n covariant vectors.

$$T \sim \vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_m \otimes \vec{u}_1 \otimes \cdots \otimes \vec{u}_n$$

$\vec{v}_i$: Contra-

$\vec{u}_j$: Co-

2.2. Basis vectors

Just as in Euclidean space, we can introduce basis vector $\vec{e}_\alpha$ $(\alpha = 0, 1, 2, 3)$

$$\vec{e}_0 \xrightarrow[\mathcal{O}]{} (1, 0, 0, 0)$$

$$\vec{e}_1 \xrightarrow[\mathcal{O}]{} (0, 1, 0, 0)$$

$$\vec{e}_2 \xrightarrow[\mathcal{O}]{} (0, 0, 1, 0) \qquad \text{ie } (\vec{e}_\alpha)^\beta = \delta_\alpha{}^\beta \qquad (2.2.1)$$

$$\vec{e}_3 \xrightarrow[\mathcal{O}]{} (0, 0, 0, 1)$$

then a vector $\vec{A} \xrightarrow[\mathcal{O}]{} (A_0, A_1, A_2, A_3)$ can be expressed as

$$\vec{A} = A^\alpha \vec{e}_\alpha \qquad (2.2.2)$$

magnitude square $\left(\|\vec{v}\|^2 \triangleq \eta_{\alpha\beta} v^\alpha v^\beta \right)$

Note that $\vec{e}_\alpha$ are "unit" vectors: $\begin{cases} \|e_0\|^2 = -1 & \leftarrow \text{time like} \\ \|e_i\|^2 = +1 & \leftarrow \text{spacelike} \end{cases}$ (2.2.3)

Note that we have written the indices of $\vec{e}_\alpha$ downstairs.

<u>Transformation of basis vectors:</u> $\vec{e}_\alpha \to \vec{e}'_\alpha = \Lambda_\alpha{}^\beta \vec{e}_\beta$ (2.2.4)

pf. when viewed in $\mathcal{O}$, $\vec{A} = A^\alpha \vec{e}_\alpha$

when viewed in $\mathcal{O}'$, $\vec{A} = A'^\alpha \vec{e}'_\alpha$

it is just the same vector, $\therefore A^\alpha \vec{e}_\alpha = A'^\alpha \vec{e}'_\alpha$

Now $A'^\alpha = \Lambda^\alpha{}_{\beta'} A^\beta$

$\therefore A^\alpha \vec{e}_\alpha = \Lambda^\beta{}_\gamma A^\gamma \vec{e}'_\beta$

$\Rightarrow \vec{e}_\alpha = \vec{e}'_\beta \Lambda^\beta{}_\alpha$

Multiply by $\Lambda_\gamma{}^\alpha$: $\Lambda_\gamma{}^\alpha \vec{e}_\alpha = \vec{e}'_\beta \Lambda^\beta{}_\alpha \Lambda_\gamma{}^\alpha = \vec{e}'_\gamma$ //

2.3 Basic manipulations of tensors

- There are several ways of forming tensors out of other tensors:

1) linear combinations: if $R^{\alpha}{}_{\beta}$, $S^{\alpha}{}_{\beta}$ are tensors, then

$$T^{\alpha}{}_{\beta} \triangleq a R^{\alpha}{}_{\beta} + b S^{\alpha}{}_{\beta} \quad \text{is also a tensor.}$$

pf.
$$\begin{aligned} T'^{\alpha}{}_{\beta} &= a R'^{\alpha}{}_{\beta} + b S'^{\alpha}{}_{\beta} \\ &= a \Lambda^{\alpha}{}_{\gamma} \Lambda_{\beta}{}^{\delta} R^{\gamma}{}_{\delta} + b \Lambda^{\alpha}{}_{\gamma} \Lambda_{\beta}{}^{\delta} S^{\gamma}{}_{\delta} \\ &= \Lambda^{\alpha}{}_{\gamma} \Lambda_{\beta}{}^{\delta} (a R^{\gamma}{}_{\delta} + b S^{\gamma}{}_{\delta}) \\ &= \Lambda^{\alpha}{}_{\gamma} \Lambda_{\beta}{}^{\delta} T^{\gamma}{}_{\delta} \end{aligned}$$

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2) Direct product: if $A^{\alpha}{}_{\beta}$, B^{γ} are tensors, then

$$T^{\alpha}{}_{\beta}{}^{\gamma} \triangleq A^{\alpha}{}_{\beta} B^{\gamma} \quad \text{is a tensor.}$$

3) Contraction: Setting an upper & lower index equal and summing it over 0 to 3 yields a tensor with two indices absent.

e.g. $T^{\alpha\gamma} \triangleq T^{\alpha}{}_{\beta}{}^{\gamma\beta}$

pf
$$T'^{\alpha\gamma} = T'^{\alpha}{}_{\beta}{}^{\gamma\beta} = \Lambda^{\alpha}{}_{\delta} \underbrace{\Lambda_{\beta}{}^{\varepsilon} \Lambda^{\gamma}{}_{\zeta} \Lambda^{\beta}{}_{\kappa}}_{\delta^{\varepsilon}{}_{\kappa}} T^{\delta}{}_{\varepsilon}{}^{\zeta\kappa}$$

$$= \Lambda^{\alpha}{}_{\delta} \Lambda^{\gamma}{}_{\zeta} T^{\delta\zeta}$$

//

4) Differentiation: The derivative of any tensor is ~~one~~ a tensor with one additional index.

e.g. $T_{\alpha}{}^{\beta\gamma} \triangleq \dfrac{\partial}{\partial x^{\alpha}} T^{\beta\gamma}$

since
$$T_{\alpha}{}^{\beta\gamma} \to T'_{\alpha}{}^{\beta\gamma} = \frac{\partial}{\partial x'^{\alpha}} T'^{\beta\gamma} = \Lambda_{\alpha}{}^{\delta} \frac{\partial}{\partial x^{\delta}} \Lambda^{\beta}{}_{\varepsilon} \Lambda^{\gamma}{}_{\zeta} T^{\varepsilon\zeta} = \Lambda_{\alpha}{}^{\delta} \Lambda^{\beta}{}_{\varepsilon} \Lambda^{\gamma}{}_{\zeta} T_{\delta}{}^{\varepsilon\zeta}$$

- There are three tensors whose components are the same in all coordinate system:

1) Minkowski tensor $\eta_{\alpha\beta}$ & Kronecker delta $\delta_\alpha{}^\beta$

The definition of Lorentz transformation tells us that $\eta_{\alpha\beta}$ is a covariant tensor:

$$\eta_{\alpha\beta} = \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \eta_{\mu\nu}$$

Similarly, $\eta^{\alpha\beta}$ is a contravariant tensor.

We can take scalar product $\eta_{\alpha\beta}\,\eta^{\beta\gamma} = \delta_\alpha{}^\gamma$

$\therefore$ $\delta_\alpha{}^\gamma$ is a tensor also.

2) Levi-Civita tensor

$$\varepsilon^{\alpha\beta\gamma\delta} \triangleq \begin{cases} 1 & \text{if } \alpha\beta\gamma\delta \text{ is an } \underline{\text{even}} \text{ permutation of } 0123 \\ -1 & \text{if } \quad " \quad " \quad \underline{\text{odd}} \text{ permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

permutation:

$\cdots ij \cdots$

$\cdots ji \cdots$

adjacent pair of objects

Note that $\varepsilon^{\varepsilon\xi\kappa\mu} \Lambda^\alpha_\varepsilon \Lambda^\beta_\xi \Lambda^\gamma_\kappa \Lambda^\delta_\mu \propto \varepsilon^{\alpha\beta\gamma\delta}$

Since LHS is odd under any single permutations of the indices $\alpha\beta\gamma\delta$.

To find the proportional constant, set $\alpha\beta\gamma\delta = 0123$,

LHS $= \det\Lambda = +1$ (proper Lorentz transf.)

$\therefore$ proportional constant $= +1$

and $\varepsilon^{\varepsilon\eta\kappa\mu} \Lambda^\alpha_\varepsilon \Lambda^\beta_\eta \Lambda^\gamma_\kappa \Lambda^\delta_\mu = \varepsilon^{\alpha\beta\gamma\delta}$

Hence $\varepsilon^{\alpha\beta\gamma\delta}$ is a tensor.

3) The zero tensor (trivial)

$$T^{\alpha_1 \cdots \alpha_n}{}_{\beta_1 \cdots \beta_m} = 0$$

2.4 Scalar product (using a metric)

Def Magnitude ~~(square)~~ of a vector $\vec{A}$, $\vec{A}^2 \triangleq A_\mu A^\mu = -A_0^2 + A_1^2 + A_2^2 + A_3^2$ (2.4.1)

it is guaranteed that $\vec{A}^2$ is Lorentz inv.

pf. A_μ, A^μ transformed oppositely. //

The magnitude does not have to be positive.

$$\begin{cases} \vec{A}^2 > 0 : & \text{spacelike} \\ \vec{A}^2 < 0 : & \text{time like} \\ \vec{A}^2 = 0 : & \text{lightlike or null} \end{cases} \quad (2.4.2)$$

Note that $\vec{A}^2 = 0$ does not imply $A^\alpha = 0$ as in Euclidean Space.

Def. Scalar product of two vector

$$\vec{A}\cdot\vec{B} \triangleq A_\mu B^\mu = -A_0 B_0 + A_1 B_1 + A_2 B_2 + A_3 B_3 \quad (2.4.3)$$

Two vectors are orthogonal if $\vec{A}\cdot\vec{B} = 0$.

Note that $\vec{A}\cdot\vec{B} = 0$ does not mean $\vec{A}$ & $\vec{B}$ are orthogonal!

eg. A null vector is orthogonal to itself.

Tetrad: A tetrad is made up of 4 mutually orthogonal vectors.

A orthonormal tetrad is ~~made of~~ $\{\vec{e}_\alpha\}_{\alpha=0,1,2,3}$ a tetrad whose vectors are of unit magnitude.

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \quad (2.4.4)$$

2.5 The four velocity & four momentum

- Associated with the worldline of a particle $x^\mu = x^\mu(\lambda)$, one can define the 4-velocity

$$U^\mu \triangleq \frac{dx^\mu}{d\tau}, \quad (2.5.1) \qquad d\tau \triangleq \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda, \text{ ie } d\tau^2 = -ds^2$$

$d\tau$ is called the proper time. (2.5.2)

rmk: only for $ds^2 \neq 0$

By definition, U^μ is tangential to the worldline.
For a particle with a constant velocity,
In the frame where the particle is at rest, it is

$$d\tau = dt$$
$$dx^\mu = (dt, 0, 0, 0)$$

$$\therefore U^\mu = (1, \underbrace{0, 0, 0}_{\text{at rest !}}) \qquad (2.5.3)$$

In general, for an accelerated particle, there is no inertial frame in which it is always at rest. However there is an inertial frame which momentarily has the same velocity as the particle. This frame is called the momentarily comoving reference frame (MCRF).

- We define the 4-momentum for a particle of mass m to be:

$$\vec{p} \triangleq m\vec{U}$$

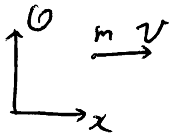
$$p^\mu = mU^\mu$$

In a frame $\mathcal{O}$,

$$\vec{p} \xrightarrow[\mathcal{O}]{} (E, \vec{p})$$ where $p^0 \triangleq E$ is called the energy of the particle in frame $\mathcal{O}$.

<u>Example:</u> A particle of mass m moves with velocity $\vec{v}$ in the x-direction of frame O.

What are the 4-velocity & 4-momentum?



$$p^\mu = m U^\mu = m \frac{dx^\mu}{d\tau}$$

$$d\tau = \sqrt{dt^2 - d\vec{x}^2} = dt\sqrt{1-v^2}$$

$$\therefore\ p^\mu = \gamma m \frac{dx^\mu}{dt} = (\gamma m, \vec{p})$$

where $\begin{cases} \vec{p} = \gamma m \vec{v} & \leftarrow \text{relativistic momentum.} \\ E = p^0 = \gamma m & \leftarrow \text{relativistic energy} \end{cases}$ (2.5.4)

For small $\vec{v}$, $\begin{cases} \vec{p} = m\vec{v} + O(v^3) \\ E = m + \frac{1}{2} m v^2 + O(v^4) \end{cases}$ (2.5.5)

Hence $E - m$ is sometimes called the kinetic energy.

Note that (2.4.1) & (2.4.2) implies that $\vec{U}\cdot\vec{U} = -1$

Conservation of four-momentum:

In Newtonian mechanics, we have conservation of energy & momentum.

Since $p^\mu = \underbrace{(E, \vec{p})}$, it is natural to postulate a relativistic ~~~~ conservation law

energy + momentum combined as a single object

of 4-momentum p^μ:

(p^μ) is conserved.

This postulate has been verified experimentally (e.g. nuclear reaction)

NB. In modern field theory, there is a connection between conservation law and symmetry. This is called the Noether theorem.

The conservation of p^μ is a result of spacetime translation symmetry:

$$x^\mu \to x^\mu + a^\mu.$$

Locally:

$$\partial^\mu T_{\mu\nu} = 0$$

$$P_\mu = \int T_{0\mu} d^3x$$

$$\dot{P}_\mu = 0$$

Q what about $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ Lorentz transf.?

Ans. The Lorentz symmetry leads to angular momentum conservation

$$\partial_\lambda \mathcal{M}^{\mu\nu\lambda} = 0 \qquad \mathcal{M}^{\mu\nu\lambda} \triangleq x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$$

$$J_{\mu\nu} = \int d^3x\, \mathcal{M}_{\mu\nu 0}$$

$$\dot{J}_{\mu\nu} = 0$$

$J^{\alpha\beta}$ is called the total angular mom.

• Note that

$$J^{0i} = t p^i - \int x^i T^{00} d^3x$$

In center of energy frame (i.e. $\int x^i T^{00} = 0$) and at $t=0$, can make $J^{0i} = 0$.

Thus it has no direct physical concept.

Nevertheless, these are needed to give a tensor $J^{\alpha\beta}$ and to define spin S_α.

• Note that under $x^\alpha \to x^\alpha + a^\alpha$,

$$J^{\alpha\beta} \to J'^{\alpha\beta} = J^{\alpha\beta} + a^\alpha p^\beta - a^\beta p^\alpha$$

So $J^{\alpha\beta}$ includes the orbital angular mom.

But it has more. To see this, define the spin (vector)

$$S_\alpha \triangleq \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} U^\delta$$

It is invariant under translation.

Moreover, in the CM frame, $(U^0 = 1, U^i = 0)$,

$S_1 = J^{23}$ etc, $S_0 = 0$

$\therefore$ C... think of S_α as the internal angular m...

2-6 photons

For photons, it always move with speed of light, hence $d\vec{x}^2 = 0$

& $ds^2 = -d\tau^2 = 0$

So the 4-velocity cannot be defined!

This can also be understood from the fact that there is no MCRF for a photon (otherwise we would have an $\vec{e}_0$ in this frame and can use it to define a 4-velocity)

Although, one cannot define a 4-velocity, nevertheless a photon still has its 4-mom.

$p^\mu = (p^0, \vec{p}) = (E, \vec{p})$.

Note that (2.5.4) for massive particles gives $\dfrac{\vec{P}}{E} = \vec{v}$

This must hold for a photon also ($|\vec{v}| = 1$). Hence $|\vec{p}| = \vec{E}$ (2.6.1)

This also means that $p^\mu p_\mu = 0$ (null).

eg. Doppler effect.

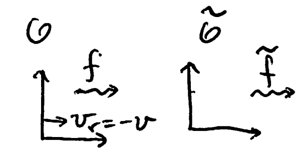
Consider a photon with freq f in frame $\mathcal{O}$ and moves in x-direction.

In $\tilde{\mathcal{O}}$ (velocity v in x-direction relative to $\mathcal{O}$), the photon energy is

$$\tilde{E} = \gamma(E - P_x v) = \gamma\, E(1-v) = E \cdot \sqrt{\frac{1-v}{1+v}}$$

Now Q.M gives $E = hf$, therefore

$$\tilde{f}/f = \sqrt{\frac{1-v}{1+v}} \qquad (2.6.2)$$



This is precisely the Doppler effect. see eqn (1.8-2) on P.20 with $v_r = -v$

2.7 3 Misc relations

1° In the MCRF, 4-velocity takes the form $U^\mu = (1,0,0,0)$

Pf. $U^\mu = \frac{dx^\mu}{d\tau}$

In the MCRF, $d\vec{x} \longrightarrow (dt, 0, 0, 0)$

$\therefore U^\mu = (1,0,0,0) = (\vec{e}_0)_{MCRF}$

2° We can define 4-acceleration as $\vec{a} = \frac{d\vec{U}}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$

Since $\vec{u}\cdot\vec{u} = -1 \Rightarrow \vec{u}\cdot\frac{d\vec{u}}{d\tau} = 0 = \vec{u}\cdot\vec{a}$ (2.7.1)

$\therefore \vec{a}$ is orthogonal to $\vec{U}$.

Now in MCRF, $\vec{u} = (1,0,0,0)$

$\therefore \vec{a} = (0, a_1, a_2, a_3)$ (2.7.2)

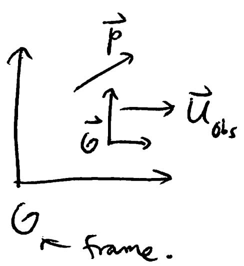
3° Recall that for a particle of mass m, $\vec{p} = m\vec{u}$ satisfied

$$\vec{p}\cdot\vec{p} = -m^2$$
$$\shortparallel$$
$$-E^2 + p^2$$

Suppose an observer $\bar{O}$ moves with four-velocity $\vec{u}_{obs}$, not necessarily equal to the particle's 4-velocity. Then

$\vec{p}\cdot\vec{u}_{obs} = \vec{p}\cdot\vec{e}_0 = -E$ (2.7.3)

This relation says that the energy of a particle relatives to a observer $\bar{O}$ is given by the projection of $\vec{p}$ onto $-\vec{u}_{obs}$.



The relation (2.7.3) is computed in the frame O where $\vec{p}$ + $\vec{u}_{obs}$ were measured. Hence (2.7.3) is frame invariant.