3. Tensor analysis in special relativity

3.1 Def of tensors

Def: An abstract definition of a tensor is a linear map from the direct product vector space $V \otimes V \otimes \cdots \otimes V \otimes \tilde{V} \otimes \cdots \otimes \tilde{V}$ to real (or complex number).

$$T : \underbrace{V \otimes V \otimes \cdots \otimes V}_{N} \otimes \underbrace{\tilde{V} \otimes \cdots \otimes \tilde{V}}_{M} \to \mathbb{C} \qquad (3.1.1)$$

This is called a tensor of type $\binom{M}{N}$.

Here V = Vector space of contravariant vector

$\tilde{V}$ = " " " covariant vector or 1-form. = Dual vector space of V

Elements in $\tilde{V}$ is called a dual vector.

- By def., a dual vector is a linear map that maps a vector to a scalar:

$$\tilde{v} \in \tilde{V} : V \to \mathbb{C}$$

$$\tilde{v} \mapsto \tilde{v}(v) \qquad (3.1.2)$$

The mapping $\tilde{v}(v)$ is bilinear. One can show that $(\tilde{\tilde{V}}) = V$.

Useful to denote $\tilde{v}(v)$ as $(\tilde{v}; v)$ or $\langle \tilde{v} | v \rangle$, called inner product.

Def. The set of all linear functions on V form a vector space, called the dual vector space $\tilde{V}$.

NB. linearity means $f(\alpha \vec{v}_1 + \beta \vec{v}_2) = \alpha f(\vec{v}_1) + \beta f(\vec{v}_2)$.

example: The "metric tensor" defined by

$$g(\vec{A}, \vec{B}) = \underset{\substack{\uparrow \\ \text{scalar product}}}{\vec{A} \cdot \vec{B}} = A^{\mu} g_{\mu\nu} B^{\nu} \qquad (3.1.3)$$

components of metric tensor

is a tensor of type $\binom{0}{2}$.

Def. The components of a tensor of type $\binom{M}{N}$ in a frame $\mathcal{O}$ are the values of the tensor when evaluated on the basis vectors $\{\vec{e}_\alpha\}$ and the dual basis vectors $\{\tilde{\omega}^\alpha\}$ of the frame $\mathcal{O}$.

NB. Tensor is coord/frame indep., but its components are coord./frame dependent.

NB. we will use the symbol $\sim$ to denote a 1-form.

3.2 Some more details on 1-forms : $\binom{1}{0}$ tensor

Consider $\vec{A}$ vector, $\tilde{p}$ 1-form, and $\{\vec{e}_\alpha\}$ the basis of a frame $\mathcal{O}$.

The components A^α of $\vec{A}$ wrt. $\mathcal{O}$ is defined by $\vec{A} \triangleq A^\alpha \vec{e}_\alpha$. (3.2.1)

The components $\tilde{p}_\alpha$ of $\tilde{p}$ wrt $\mathcal{O}$ is defined by $\tilde{p}(\vec{e}_\alpha) \triangleq p_\alpha$.

Since the set of all 1-form is a vector space $\tilde{V}$, we can ~~use~~ pick any set of four linearly independent 1-forms as a basis, Denote it as $\{\tilde{\omega}^\alpha\}$ s.t any $\tilde{p}$ can be expanded as $\tilde{p} = p_\alpha \tilde{\omega}^\alpha$

However, suppose we have a basis $\{\vec{e}_\alpha\}$ already for the frame $\mathcal{O}$.

It is natural to consider a dual basis s.t.

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \qquad (3.2.2)$$

This implies that for any, $\tilde{p} = p_\alpha \tilde{\omega}^\alpha$

$$\vec{A} = A^\alpha \vec{e}_\alpha$$

we have $\tilde{p}(\vec{A}) = p_\alpha A^\beta \tilde{\omega}^\alpha(\vec{e}_\beta) = p_\alpha A^\alpha$ (3.2.3)

a contraction!

- Under a coord change, $\mathcal{O} \to \mathcal{O}'$ $\quad \vec{e}'_\beta = \vec{e}_\alpha \Lambda^\alpha{}_\beta \qquad (3.2.4)$

This implies the transformation

$$\tilde{\omega}'^\alpha = \Lambda^\alpha{}_\beta \, \tilde{\omega}^\beta \qquad (3.2.5)$$

for the dual basis.

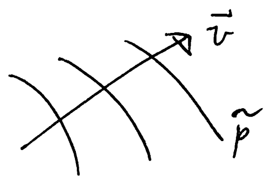
- Picture of 1-form

For vector, we usually represent it with an arrow, whose length corresponds to the magnitude of the vector. $\vec{v}$

For 1-form, we can represent it by a series of surfaces.

$\tilde{p}$

Such that $\tilde{p}(\vec{v})$ is the number of surfaces that the arrow $\vec{v}$ crosses.



Note that a 1-form does not define a unique direction since it is not a vector.

- An example of a one-form is the gradient of a scalar function ϕ, defined by components $(\tilde{d}\phi)_\alpha \triangleq \partial_\alpha \phi = \dfrac{\partial \phi}{\partial x^\alpha}$

When evaluated on the tangent vector $v^\alpha \triangleq \dfrac{dx^\alpha}{d\tau}$ of a world line $x^\alpha(\tau)$,

we have $\vec{v}(\tilde{d}\phi) = v^\alpha \partial_\alpha \phi = \dfrac{dx^\alpha}{d\tau}\dfrac{\partial \phi}{\partial x^\alpha} = \dfrac{d}{d\tau}\phi(x^\alpha(\tau))$

= rate of change of the function ϕ along the curve $x^\alpha(\tau)$.

- It is sometimes useful to talk about a <u>normal 1-form</u>.

Given a surface, a normal vector is one which is <u>orthogonal</u> to the surface.

need a metric to define!

$$\vec{N} \cdot \vec{T} = 0 \qquad \forall \vec{T} \text{ tangential to the surface.}$$

on the other hand, to define a normal 1-form, there is no need of a metric:

$$\tilde{\omega}(\vec{T}) = 0 \qquad \forall \ \vec{T} \text{ tangential to the surface.}$$

Moreover, we say a normal one-form is outward if

$$\tilde{\omega}(\vec{v}) > 0 \qquad \text{for all vectors } \vec{v} \text{ which point outwards from the surface}$$

3.3 Some more details of the $\binom{0}{2}$ tensors

we have seen that the metric tensor is a $\binom{0}{2}$ tensor.

Another example is to simply take product of two 1-forms: $\tilde{p} \otimes \tilde{q}$.

called Cartesian product, outer

$$\tilde{p} \otimes \tilde{q}\,(\vec{A}, \vec{B}) \triangleq \tilde{p}(\vec{A})\,\tilde{q}(\vec{B}) \qquad (3.3.1)$$

Note that $\tilde{p} \otimes \tilde{q}$ & $\tilde{q} \otimes \tilde{p}$ are different $\binom{0}{2}$ tensors.

<u>Thm</u>. A general $\binom{0}{2}$ tensor can be written as a sum of outer products.

<u>Pf</u>. Let $\tilde{p}$ be a $\binom{0}{2}$ tensor with components $p_{\alpha\beta} \triangleq \tilde{p}(\vec{e}_\alpha, \vec{e}_\beta)$.

On arbitrary vectors $\vec{A}, \vec{B}$, we have $\tilde{p}(\vec{A}, \vec{B}) = p_{\alpha\beta} A^\alpha B^\beta$.

The question is how is $\tilde{p}$ & $p_{\alpha\beta}$ related?

~~take the~~

In general, we can assume that $\tilde{p} = p_{\alpha\beta}\, \tilde{\omega}^{\alpha\beta}$ for some tensor $\tilde{\omega}^{\alpha\beta}$.

then $\tilde{p}(\vec{A},\vec{B}) = p_{\alpha\beta}\,\tilde{\omega}^{\alpha\beta}(\vec{A},\vec{B}) \overset{!}{=} p_{\alpha\beta} A^{\alpha} B^{\beta}$

$$\| $$

$$p_{\alpha\beta} A^{\gamma} B^{\delta}\, \tilde{\omega}^{\alpha\beta}(\vec{e}_{\gamma}, \vec{e}_{\delta})$$

This holds for arb A^{α}, B^{β}, so need $\tilde{\omega}^{\alpha\beta}(\vec{e}_{\gamma}, \vec{e}_{\delta}) = \delta^{\alpha}_{\gamma}\,\delta^{\beta}_{\delta}$

$\therefore \tilde{\omega}^{\alpha\beta}$ is a tensor whose value is just the product of two basis one-forms:

$$\tilde{\omega}^{\alpha\beta} = \tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta}$$

$\therefore$ the tensors $\tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta}$ are a basis for all $\binom{0}{2}$ tensors and

$$\tilde{p} = p_{\alpha\beta}\, \tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta}$$

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- We call a $\binom{0}{2}$ tensor symmetric if $\tilde{p}(\vec{A},\vec{B}) = \tilde{p}(\vec{B},\vec{A}) \quad \forall \vec{A},\vec{B}$

In terms of components, $p_{\alpha\beta} = p_{\beta\alpha}$ (3.3.2)

Similarly, $\tilde{p}$ is antisymmetric if $\tilde{p}(\vec{A},\vec{B}) = -\tilde{p}(\vec{B},\vec{A}) \quad \forall \vec{A},\vec{B}$

$$p_{\alpha\beta} = -p_{\beta\alpha} \qquad (3.3.3)$$

In general, given a $\binom{0}{2}$ tensor, it may not be symmetric or antisymmetric,

but we can always define its symmetric and antisymmetric parts as

$$\begin{cases} h^{(S)}_{\alpha\beta} \triangleq \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) \\ h^{(AS)}_{\alpha\beta} \triangleq \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) \end{cases} \qquad (3.3.4)$$

Note that $h_{\alpha\beta} = h^{(S)}_{\alpha\beta} + h^{(AS)}_{\alpha\beta}$

The factor $\frac{1}{2}$ in $h^{(S)}_{\alpha\beta}$ is defined s.t. for symmetric tensor, it is equal to its symmetric part.

3.4 Identification of vectors & 1-forms using $\binom{0}{2}$ or $\binom{2}{0}$ tensors

- In general, vectors & 1-forms are dual to each other in the sense of linear vector space. In the presence of a $\binom{0}{2}$ tensor, one can map vector to 1-form or vice-versa directly.

Let g be a $\binom{0}{2}$ tensor, it has two arguments: $g(\cdot,\cdot)$

If we fix $\vec{v} \in V$, then $g(\vec{v},\cdot)$ is a functional on V and we may call this 1-form as $\tilde{v}$:

hence a 1-form

$$\tilde{v}(\vec{w}) \triangleq g(\vec{v},\vec{w}) \qquad (3.4.1)$$

Components of $\tilde{v}$ can be worked out as follows:

$$v_\alpha \triangleq \tilde{v}(\vec{e}_\alpha) = g(\vec{v},\vec{e}_\alpha) = v^\beta g(\vec{e}_\beta,\vec{e}_\alpha) \qquad \vec{v} = v^\beta \vec{e}_\beta$$

$$\therefore v_\alpha = v^\beta g_{\beta\alpha} \qquad (3.4.2)$$

- Inversely, we could also go from a 1-form to a vector.

Let $\tilde{v} \in \tilde{V}$ be a 1 form, and $\tilde{g}$ a $\binom{2}{0}$ tensor,

then $\tilde{g}(\tilde{v},\cdot)$ is a linear functional on $\tilde{V}$.

Denote this by $\vec{v}$, then

$$\vec{v}(\tilde{\omega}) \triangleq \tilde{g}(\tilde{v},\tilde{\omega}) \qquad (3.4.3)$$

In terms of components,

$$v^\alpha = g^{\alpha\beta} v_\beta \qquad \text{where } g^{\alpha\beta} \triangleq \tilde{g}(\omega^\alpha,\omega^\beta) \qquad (3.4.4)$$

- In general, $g, \tilde{g}$ are tensors of type $\binom{0}{2}$ and $\binom{2}{0}$.

In specific problem, the system may admit a metric

a $\binom{0}{2}$ tensor which is non-degenerate.

In this case, as g is non-degenerate, it's inverse exists and is a $\binom{2}{0}$ tensor

In components, we denote the metric as $g_{\alpha\beta}$ and the inverse as $g^{\alpha\beta}$.

3.5 Basic manipulations of tensors

refer to P.26 & P.27 (§ 2.3)