4. Physics of Special relativity in terms of tensorial notation

4.1 Particle dynamics ("Newton" laws)

Newtonian mechanics is good for low velocities. But it is not compatible with the principle of relativity (eg. its prediction of addition of velocity is wrong), so it has to be modified.

Consider a particle with a worldline $x^\alpha(\tau)$. Define the following quantity

$$f^\alpha \triangleq m \frac{d^2 x^\alpha}{d\tau^2} \qquad \text{—(4.1.1)}$$

Called the relativistic force. We can relate it to the usual Newtonian force by noting its properties:

1°. In the MCRF, particle is momentarily at rest $\Rightarrow d\tau = dt$

$$\therefore f^0 = 0 \quad \text{and} \quad f^{\dot{\alpha}} = F^{\dot{\alpha}} \quad \leftarrow \text{nonrelativistic force} \qquad \text{—(4.1.2)}$$

2°. f^α is a four vector: $f^\alpha \to f'^\alpha = \Lambda^\alpha{}_\beta f^\beta$ —(4.1.3)

Now suppose our particle has velocity $\vec{v}$ at time t_0.

Introduce a new coord system x'^α defined by

$$x^\alpha = \Lambda^\alpha{}_\beta(\vec{v})\, x'^\beta \qquad \text{—(4.1.4)}$$

It is clear that the particle is at rest in the frame x'^β. Thus the frame x'^α constructed as in (4.1.4) is the MCRF.

Therefore according to (4.1-3), $f^{\alpha} = \Lambda^{\alpha}{}_{\beta}(\vec{v}) F^{\beta}$ $\qquad F^{\beta} = (F^{0}, F^{i}) = (0, \vec{F})$

More explicitly,

$$\begin{cases} \vec{f} = \vec{F} + (\gamma - 1)\, \vec{v}\, \dfrac{(\vec{v}\cdot\vec{F})}{v^2} \\ \\ f^{0} = \gamma\, \vec{v}\cdot\vec{F} = \vec{v}\cdot\vec{f} \end{cases} \qquad (4.1\text{-}5)$$

$\vec{v}$ = instantaneous velocity.

Given that we know how to calculation f^{α} now, we could solve (4.1.1) and calculate $x^{\alpha}(\tau)$, then eliminate τ to get $\vec{x}(t)$.

However this is an initial value problem and the initial values of $\dfrac{dx^{\alpha}}{d\tau}$ must be choosen carefully such that $d\tau$ is really the proper time. ie.

$$-1 = \eta_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} \qquad (4.1\text{-}6)$$

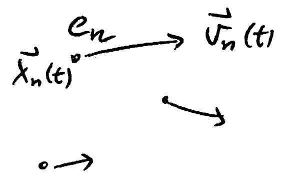
4.2 Current and densities of Charges

Suppose we have a system of particles with position $\vec{x}_n(t)$ and charges e_n.

we can define the current and charge densities as:

$(\vec{J} = q\vec{v})$

$$\begin{cases} \vec{J}(\vec{x}, t) = \sum_n e_n\, \delta^{(3)}(\vec{x} - \vec{x}_n(t))\, \dfrac{d\vec{x}_n}{dt} \\ \\ \varepsilon(\vec{x}, t) = \sum_n e_n\, \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \end{cases} \qquad -(4.2\text{-}1)$$



$\delta^{(3)}$ is the Dirac delta function: $\int d^3\vec{y}\, f(\vec{y})\, \delta^{(3)}(\vec{x} - \vec{y}) = f(\vec{x})$ $\qquad -(4.2.2)$

We can unify $\vec{J}$, ε into a single formula by calling $\varepsilon \triangleq J^{0}$, then $J^\alpha = \sum_n e_n \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \dfrac{dx_n^\alpha}{dt}$ —(4.2.3)

- Claim: J^α is a 4-vector

To show this, we write (4.3.3) as:

$$J^\alpha_{(x)} = \int dt' \sum_n e_n \delta^{(4)}(x - x_n(t')) \frac{dx_n^\alpha(t')}{dt'}$$

where $x_n^0(t) \triangleq t$

we can cancel the differential dt' and replace it by τ,

$$J^\alpha_{(x)} = \int d\tau \sum_n e_n \delta^{(4)}(x - x_n(\tau)) \frac{dx_n^\alpha(\tau)}{d\tau}$$

Since $\delta^{(4)}(x - x_n(\tau))$ is a scalar, dx_n^α is a 4-vector, so J^α is a 4-vector.

- Note that $\nabla\cdot\vec{J} = \sum_n e_n \dfrac{\partial}{\partial x^i} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \dfrac{dx_n^i(t)}{dt}$

$$= -\sum_n e_n \underbrace{\frac{\partial}{\partial x_n^i} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \frac{dx_n^i(t)}{dt}}_{\frac{\partial}{\partial t}\delta^{(3)}(\vec{x} - \vec{x}_n(t))}$$

$$= -\frac{\partial}{\partial t} \mathcal{E} \qquad (4.2.4)$$

Conservation of Charges!

$\therefore \partial_\alpha J^\alpha = 0$

• The conservation law implies the existence of a conserved charge:

$$Q \hat{=} \int d^3x \; J^0(x)$$

$$\frac{dQ}{dt} = \int d^3x \, \frac{\partial}{\partial x^0} J^0 = -\int d^3x \; \nabla \cdot \vec{J} = 0$$

4.3 Energy - Momentum tensor

One can similarly give a definition for the density and current for the energy-mom. 4-vector p^α (in previous section, we did it for charge).

The density of p^α is defined by

$$T^{\alpha 0}(\vec{x},t) = \sum_n p_n^\alpha(t)\, \delta^3(\vec{x} - \vec{x}_n(t)) \qquad (4.3.1)$$

The current is defined by:

$$T^{\alpha i}(\vec{x},t) = \sum_n p_n^\alpha(t) \frac{dx_n^i(t)}{dt} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \qquad (4.3.2)$$

The two definitions can be united into:

$$T^{\alpha\beta}(x) = \sum_n p_n^\alpha \frac{dx_n^\beta(t)}{dt} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \qquad (4.3.3)$$

For particles, $p_n^\beta = E_n \dfrac{dx_n^\beta}{dt}$ (see (2.5.4)) (4.3.4)

and so

$$T^{\alpha\beta} = \sum_n \frac{p_n^\alpha \, p_n^\beta}{E_n} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) = \sum_n E_n \frac{dx_n^\alpha}{dt} \frac{dx_n^\beta}{dt} \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \qquad (4.3.5)$$

hence $T^{\alpha\beta} = T^{\beta\alpha}$ is symmetric.

One can similarly show that $T^{\alpha\beta}$ is a tensor.

- Conservation law:

$$\frac{\partial}{\partial x^i} T^{\alpha i}(\vec{x},t) = -\sum_n p_n^\alpha \frac{dx_n^i}{dt}\frac{\partial}{\partial x_n^i}\delta^{(3)}(\vec{x}-\vec{x}_n(t))$$

$$= -\sum_n p_n^\alpha \frac{\partial}{\partial t}\delta^{(3)}(\vec{x}-\vec{x}_n(t))$$

$$= -\frac{\partial}{\partial t} T^{\alpha 0}(\vec{x},t) + \underbrace{\sum_n \frac{dp_n^\alpha(t)}{dt}\delta^{(3)}(\vec{x}-\vec{x}_n(t))}_{G^\alpha}$$

$$\Rightarrow \quad \partial_\beta T^{\alpha\beta} = G^\alpha \qquad (4.3\text{-}6)$$

G^α is called the density of force:

$$G^\alpha(\vec{x},t) = \sum_n \delta^{(3)}(\vec{x}-\vec{x}_n(t))\frac{d\tau}{dt} f_n^\alpha(t) \qquad (4.3.7)$$

$G^\alpha = 0$ if the particles are free and so p_n^α is constant.

- Given generally a conserved $T^{\mu\nu}$, one can define another conserved tensor:

$$M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$$

$$\partial_\lambda M^{\mu\nu\lambda} = 0$$

Hence the "charge" is conserved:

$$J^{\alpha\beta} \triangleq \int d^3x \; M^{\alpha\beta 0} = -J^{\beta\alpha}$$

This is called the total angular momentum tensor.

(See p. 31 for more discussion)

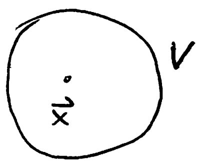
4.4 Fluid

The description of a system in terms of individual particles is fine but often too complicated since we often do not need the details of individual particles to describe a system, but a description in terms of average or bulk quantities is sufficient.

e.g. no. of particles per unit volume, density of momentum, pressure, temperature.

Generally, these properties ~~are function of~~ varies over space + time. To obtain the average quantities, one need to take an average over spatial volume V,

$$\bar{O}(\vec{x},t) \triangleq \Big(\sum_{n \in V} O(\vec{x}_n(t))\Big)\frac{1}{N} \qquad (4.4.1)$$



V is a volume element centered at $\vec{x}$. There are N particles inside V.

The volume V has to be chosen appropriately. It has to be large enough (to contain a large # of particles) so that $\bar{O}$ is given by a smooth enough function.
(differentiable)

It can't be too large as we would like to have a somewhat local description of the system.

Generally such an averaging may not exist. (e.g. rarified gas).

When it exists, we say the system has a <u>continuum approximation/description</u>.

As usual, such a description has its limitation. If we go to a smaller scale, then the continuum approx may break down.

(e.g. if we are interested in scale of physics which is ~~just a~~ ~~few~~ close to the av. molecular distance.

So when we say we have a continuum, we are implicitly assuming we work in a scale such that the continuum approx. is valid.

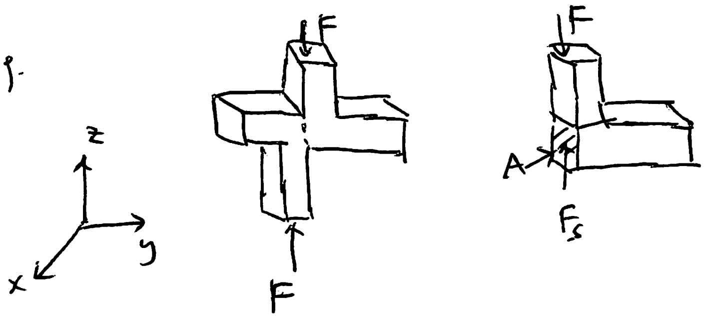
A continuum can be a <u>solid</u> or <u>fluid</u>.

Consider a continuum in static equilibrium, and apply an external force.

The force can be resolved into components normal to the surface of the continuum, called pressure and components tangential to the surface, called tensile stress/shear.

Generally, consider an area element with normal vector in the ith direction and a force in the jth direction, we get a stress tensor σ_{ij}. —(44.2)

e.g.



$$\text{Shear force} = \frac{F_s}{A} = \sigma_{zy}$$

- When a stress is applied, the continuum will normally deformed (changing shape).

Def A fluid is defined to be one which flow (adjacent elements slide along their common boundary) when a shear force is applied.

Def. A perfect fluid is one in which no shear force is supported.

The only force between adjacent fluid elements is pressure.

$T^{\mu\nu}$ for fluid.

- Generally, we can define the energy-momentum tensor as follows (w.r.t. to an arb frame):

$$T^{\alpha\beta} \doteq \text{flux of } p^{\alpha} \text{ across a surface of constant } x^{\beta} \qquad -(4.4.3)$$

As a result, $T^{00} =$ flux of energy across a $\underbrace{\text{surface } t = \text{constant}}_{\text{3-volume}}$

$= \text{energy density}$

T^{0i} = flux of energy across surface $x^i = \text{const.}$

= energy flux across the x^i surface

T^{i0} = flux of momentum across a surface $t = \text{constant}$

$= i$ momentum density

T^{ij} = flux of i momentum across j surface.

Claim. T^{ij} according to definition (4.4.3) is exactly the stress tensor σ_{ij} defined in (4.4.2).

Pf. This is simple. The flux of i-momentum across j-surface

$$= \underbrace{\text{rate of change of mom. in the } i^{th} \text{ direction}}_{F_i} \text{ per unit area } \hat{j}\text{-}$$

$$= F_i / A_j$$

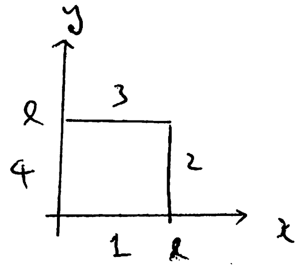
$$= \sigma_{ij}$$

//

Thm In the absense of external force, energy & momentum is conserved.

This can be expressed as $\partial_\alpha T^{\alpha\beta} = 0$ (4.4.4)

Pf. Consider a cubical fluid element of length l.



· Energy can flow in across all surfaces.

The rate of flow of energy across face (4) $= l^2 T^{0x}(x=0)$

" (2) $= -l^2 T^{0x}(x=l)$

" (1) $= l^2 T^{0y}(y=0)$

(3) $= -l^2 T^{0y}(y=l)$

∴ rate of increase of energy inside $= \frac{\partial}{\partial t}(l^3 T^{00})$

$$= l^2 \big[T^{0x}(x=0) - T^{0x}(x=l) + T^{0y}(y=0) - T^{0y}(y=l) + T^{0z}(z=0) - T^{0z}(z=l) \big]$$

$l \to 0$ $\Rightarrow$ $\frac{\partial}{\partial t} T^{00} + \partial_i T^{0i} = 0$

ie $\partial_\alpha T^{0\alpha} = 0$

Similarly, conservation of momentum implies $\partial_\alpha T^{i\alpha} = 0$

//

Number-flux N^μ.

In addition to energy & momentum, it is also useful to introduce the notion of a number density. Consider a certain fluid element and let n be the number density in the MCRF of the element.

Then in general the number density in a frame in which the element has velocity v $= \frac{n}{\sqrt{1-v^2}}$ (4.4.5)

It is also useful to introduce flux, which is defined as the number of particle across a unit area per unit time. It is clear that

$$(\text{flux})^i = \frac{v^i n}{\sqrt{1-v^2}} \tag{4.4.6}$$

It is useful to combine the above two to define the quantity

$$N^\alpha = n\left(\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}}\right) = n U^\alpha \tag{4.4.7}$$

It is clear this is a 4-vector.

- Apart from conservation of energy-momentum, it may happen that during the flow of the fluid, the number of particle in a fluid is conserved. The local change in the number density can only be compensated by an inflow of particles as flux. ie.

$$\frac{\partial}{\partial t} N^0 = - \partial_i N^i$$

or $\quad \partial_\alpha N^\alpha = 0$

First law of thermodynamics

The first law reads $\quad \Delta E = \Delta Q - p \Delta V$

Consider an element with N particles,

Assume particle number conserves, then $\quad V = \frac{N}{n}$

$$\Delta V = -\frac{N}{n^2} \Delta n$$

Also $\quad E = \rho V = \rho \frac{N}{n} \quad \Rightarrow \quad \Delta E = \Delta \rho \frac{N}{n} - \frac{\rho N}{n^2} \Delta n$

$$\therefore \Delta Q = \Delta E + p \Delta V = \Delta \rho \frac{N}{n} - (\rho + p) \frac{N}{n^2} \Delta n$$

$$\Delta q = \frac{\Delta Q}{N} = \frac{1}{n}\left(\Delta \rho - \frac{(\rho + p)}{n} \Delta n \right) \qquad q \hat{=} \frac{Q}{N}$$

or. $\quad n \, dq = d\rho - (p + \rho) \, dn/n$

$\because \quad dq = T \, dS$

$$\therefore \quad nT \, dS = d\rho - (p + \rho) \frac{dn}{n} \qquad (4.4.8)$$

This is the first law, written in terms of p, ρ, n, variables convenient for fluid.

4.5 Examples of Fluid

In general a fluid is characterized by

$$\begin{cases} \rho = \text{energy density} \\ q^\alpha = \text{heat flux vector} \\ p = \text{pressure} \\ u^\alpha = \text{4-velocity of fluid element.} \\ \Pi^{\alpha\beta} = \text{viscous stress tensor.} \end{cases}$$

q^α & $\Pi^{\alpha\beta}$ are transverse to the worldlines,

i.e. $q^\alpha u_\alpha = 0$, $\Pi^{\alpha\beta} u_\beta = 0$

The energy-momentum tensor reads

$$T^{\alpha\beta} = (\rho + p) u^\alpha u^\beta + p\,\eta^{\alpha\beta} + (u^\alpha q^\beta + u^\beta q^\alpha) + \Pi^{\alpha\beta} \qquad (4.4.9)$$

Special cases:

1) A perfect fluid is one with vanishing q^α, $\Pi^{\alpha\beta}$.

$$T^{\alpha\beta} = (p+\rho) u^\alpha u^\beta + p\,\eta^{\alpha\beta} \qquad (4.4.10)$$

2) A dust is perfect ~~one~~ fluid without any pressure:

$$T^{\alpha\beta} = \rho\, u^\alpha u^\beta \qquad (4.4.11)$$

3) A radiation fluid is a perfect fluid with $p = \rho/3$

$$T^{\alpha\beta} = p(4u^\alpha u^\beta + \eta^{\alpha\beta}) \qquad (4.4.12)$$

Eg. 1° Perfect Fluid

Perfect fluid is one in which it looks isotropic in the MCRF of its fluid elements. Thus in its MCRF, $\tilde{T}^{\mu\nu}$ must look spherically symmetric:

$$\begin{cases} \tilde{T}^{ij} = p\,\delta_{ij} \\ \tilde{T}^{i0} = \tilde{T}^{0i} = 0 \\ \tilde{T}^{00} = \rho \end{cases} \qquad (4.4.13)$$

The coefficient ρ, p are called the proper energy density & pressure of the fluid.

Now assume the MCRF has a velocity $\vec{v}$ wrt. the lab frame x^α,

then $\quad x^\alpha = \Lambda^\alpha{}_\beta(\vec{v})\,\tilde{x}^\beta$.

When $\Lambda^\alpha{}_\beta$ is given by: $\begin{cases} \Lambda^0{}_0 = \gamma \\ \Lambda^i{}_0 = \gamma v_i = \Lambda^0{}_i \\ \Lambda^i{}_j = \delta_{ij} + v_i v_j \dfrac{\gamma - 1}{v^2} \end{cases}$

As tensor, we have

$$T^{\alpha\beta} = \Lambda^\alpha{}_\gamma(\vec{v})\,\Lambda^\beta{}_\delta(\vec{v})\,\tilde{T}^{\gamma\delta}$$

or explicitly $\quad T^{ij} = p\,\delta_{ij} + (p+\rho)\dfrac{v_i v_j}{1-v^2}$

$$T^{i0} = (p+\rho)\frac{v_i}{1-v^2}$$

$$T^{00} = \frac{\rho + p v^2}{1-v^2}$$

or in a simple eqn, $\quad T^{\alpha\beta} = p\,\eta^{\alpha\beta} + (p+\rho)\,u^\alpha u^\beta \qquad$ i.e. (4.4.10) !

• Generally, the motion of the fluid is determined by Conversation laws such as

$$\begin{cases} \text{Conservation of energy-momentum} \\ \text{Conservation of particle number.} \\ \text{and the eqn. of state.} \end{cases}$$

$$0=\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = \frac{\partial p}{\partial x^\alpha} + \frac{\partial}{\partial x^\beta}\left((p+\rho)u^\alpha u^\beta\right) \qquad (4.4.14)$$

$$0=\frac{\partial N^\alpha}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}(n u^\alpha) = \frac{\partial}{\partial t}\left(n\frac{1}{\sqrt{1-v^2}}\right) + \nabla\cdot\left(\frac{n\vec{v}}{\sqrt{1-v^2}}\right) \qquad (4.4.15)$$

<u>Lemma</u> (4.4.14) can be written as ↙ Euler eqn.

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{(1-v^2)}{p+\rho}\left[\nabla p + \vec{v}\frac{\partial p}{\partial t}\right] \qquad (4.4.16)$$

and the scalar eqn. $u^\beta\frac{\partial\sigma}{\partial x^\beta}=0$ ← σ = entropy (4.4.17)

Pf. $\alpha=0$, $\frac{\partial p}{\partial t} + \frac{\partial}{\partial x^\beta}\left((p+\rho)\gamma u^\beta\right) = 0$

$\alpha = i$, $\frac{\partial p}{\partial x^i} + \underbrace{\frac{\partial}{\partial x^\beta}\left((p+\rho)\gamma v^i u^\beta\right)}_{=\, v^i\left(-\frac{\partial p}{\partial t}\right) + (p+\rho)\gamma\, \underbrace{u^\beta \frac{\partial}{\partial x^\beta} v^i}_{=\, \gamma\left[\frac{\partial v^i}{\partial t} + v^j\frac{\partial}{\partial x^j}v^i\right]}} = 0$

$$\Rightarrow \frac{\partial v^i}{\partial t} + (\vec{v}\cdot\nabla)v^i = -\frac{1-v^2}{(p+\rho)}\left[\partial_i p + v^i\frac{\partial p}{\partial t}\right]$$

//

To obtain the scalar eqn (4.4.17), we note that law of

Thermodynamics reads

$$kT\,d\sigma = p\,d\left(\frac{1}{n}\right) + d\left(\frac{\rho}{n}\right) \qquad (4.4.18)$$

Now multiply (4.4.14) with u_α and use $0 = u_\alpha \frac{\partial u^\alpha}{\partial x^\beta}$,

we get
$$u^\alpha \partial_\alpha p - \underbrace{\frac{\partial}{\partial x^\beta}\left[(p+\rho)u^\beta\right]} = 0$$

$$\| $$

$$\frac{\partial}{\partial x^\beta}\left[\frac{p+\rho}{n}(nu^\beta)\right] \underset{(4.4.15)}{=\!=} nu^\beta \frac{\partial}{\partial x^\beta}\left(\frac{p+\rho}{n}\right)$$

$$\Rightarrow -nu^\beta \underbrace{\left[p\frac{\partial}{\partial x^\beta}\left(\frac{1}{n}\right) + \frac{\partial}{\partial x^\beta}\left(\frac{\rho}{n}\right)\right]}_{kT\frac{\partial \sigma}{\partial x^\beta}} = 0$$

$$\Rightarrow u^\beta \frac{\partial \sigma}{\partial x^\beta} = \frac{\partial \sigma}{\partial t} + (\vec{v}\cdot\nabla)\sigma = 0 \qquad (4.4.18)$$

//

Comments

i) physically, (4.4.17) is a statement that the specific entropy is constant in time at any point that moves along with the fluid. This meant $\frac{d\sigma}{d\tau} = u^\beta \frac{\partial \sigma}{\partial x^\beta} = 0$

Specific entropy is conserved and the flow is called ~~adiab~~ adiabatic.

ii) The LHS of (4.4.16) is an acceleration

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$

Thus (4.4.16) is a generalized statement of Newton 2nd law: $\vec{F} = m\vec{a}$.

Ep-2° Dust: since $u^\alpha = \gamma(1, \vec{v})$,

so in MCRF, $u^\alpha = (1, \vec{0})$ and

$$(T^{\alpha\beta})_{MCRF} = \begin{cases} (T^{00})_{MCRF} = \rho \\ (T^{0i})_{MCRF} = (T^{i0})_{MCRF} = (T^{ij})_{MCRF} = 0 \end{cases}$$

∴ In MCRF, there is simply the rest mass of the particles contributing to $T^{\alpha\beta}$.

Physically, dust consists of a system of particles whose action is given by

$$S_M = -\int \rho \sqrt{u^2}\, d^4x \sqrt{|g|} \qquad |g| = 1 \text{ here.}$$

Dust is a simple fluid where in the MCRF of its fluid elements the energy is given by the mass and there is no momentum flow.