5. Gravity as curved geometry

5.1 Equivalence Principle and gravity as geometry

In this section, we discuss why gravity can be described as a effect of curved spacetime geometry as a result of the equivalence principle. Let us first explain the equivalence principle.

- Equivalence Principle is a statement about the universality of the gravitational interaction.

The Weak Equivalence Principle (WEP) states that the gravitational mass and the inertial mass of any object are equal:

Newton 2nd law $\vec{F} = m_i \vec{a}$ $\qquad$ m_i = inertial mass = resistance of motion to applied force $\vec{F}$

Newton law of gravitation: $\vec{F}_g = -m_g \nabla \Phi$ $\quad$ (Φ: gravitational potential.)

m_g has a very different character from m_i, it is specific to grav. force.

However it has been shown first by Galileo that $m_i = m_g$. $\qquad$ (5.1.1)

(Falling objects from the tower of Pisa)

Therefore $\vec{a} = -\nabla \Phi$.

Consider the case of a constant (homogeneous & time independent) gravitational field $\vec{g}$,

then $\dfrac{d^2\vec{x}}{dt^2} = \vec{g}$.

Suppose we perform a (non-Galileo) spacetime coord. transformation:

$$\vec{x}' = \vec{x} - \frac{1}{2} g t^2 , \qquad t' = t \qquad (5.1.2)$$

then $\frac{d^2\vec{x}'}{dt^2} = 0$

$(\vec{x}', t')$ coordinate system is a free falling system (as clear from (5.1.2))

Therefore a free falling observer experience no gravity at all! ← (5.1.3)

This statement was derived for a constant $\vec{g}$. In general the gravity field is not homogeneous & could be time dependent. Nevertheless, if we restrict to such a small region of space & time that the field changes very little over the region, then we can expect an approximate cancellation & the free falling observer would still experience no gravitational force!

The <u>(strong) Equivalence Principle</u> (SEP) states that:

> At every spacetime point in an arbitrary grav. field, it is possible to choose a "locally inertial coord. system" such that within a sufficiently small region,
> locally free falling observer
>
> the <u>laws of nature</u> takes the same form as in unaccelerated Cartesian coord. systems in the absence of gravitation. (5.1.4)

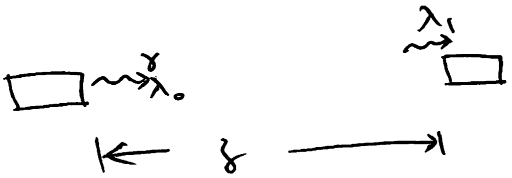
<u>N.B.</u> 1° The SEP refers to all laws of nature.
The WEP refers to the laws of motion of freely falling particles.

2° The experiments of Eötvös (cf. Weinberg §1.2) provides experimental support of WEP and indirect evidence for the SEP.

3° Sometime we refer to WEP as Galileo's equivalence Principle
SEP as Einstein Equivalence Principle.

• A consequence of EEP : gravitational red shift.

Consider two boxes, a distance z apart, each moving with a constant acceleration a.



At time t_0, a photon of wavelength λ_0 was emitted.

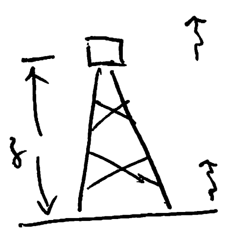
The boxes remains a constant distance z apart.

The photon reaches the leading box after a time $\Delta t = z/c$ in the ref. frame of the boxes. In this time, the boxes will have picked up a vel. $\Delta v = a \Delta t = a z/c$.

Therefore the photon reaching the leading box will be redshift by the conventional Doppler effect

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2} \qquad (5.1.5)$$

Now according to EEP, the same thing would happen to a uniform grav. field.



Imagine a tower of height z in a constant gravity field a_g

According to EEP, we should detect a grav. red shift:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2} \qquad (5.1.6)$$

verified first by Pound, Rebka (1960).

NB. For a non-uniform grav. field, the formula is replaced by

$$\frac{\Delta\lambda}{\lambda_0} = \frac{1}{c^2}\int a_g\, dz = \frac{1}{c^2}\int \nabla\Phi\, dz = \frac{1}{c^2}\Delta\Phi \qquad (5.1.7)$$

True also in GR in the weak field approx.!

since we use the Newtonian potential!

- Implication : gravity as curved manifold

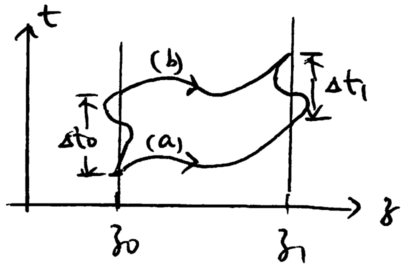
The grav. redshift leads to an argument that we should consider spacetime as curved!

Consider the same experiment as above. Suppose a physicist on the ground emits a beam of light with wavelength λ_0 at height z_0, which travels to the top of the tower at height z_1.

At z_0, the time between one wavecycle is given by $\Delta t_0 = \lambda_0 / c$

At z_1, " " " " " $\Delta t_1 = \lambda_1 / c$

For a constant grav. field, the path through spacetime followed by the leading (a) and trailing (b) edge of the single wave must be congruent.



If the geometry were flat, then $\Delta t_0 = \Delta t_1$

Euclidean

However grav. red shift implies that $\Delta t_0 < \Delta t_1$

$\therefore$ spacetime must be <u>curved</u>!

Summarizing, Equivalence principle tells us that locally one can always find a local inertial frame in which the laws of physics look like those of SR (i.e. spacetime is Minkowski). But in an extended region where gravity cannot be ignored, spacetime must be curved.

The appropriate mathematical concept for ~~descri~~ describing this situation is manifold.

5.2 Gravitational force

Consider a particle moving under the influence of pure gravitational force.

We would like to determine its equation of motion.

Let x^α be the lab frame and ξ^α be the freely falling coord. system (local inertial frame)

According to the EEP, there is no grav force in the ξ^α frame.

Hence the equation of motion is a straight line:

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \qquad (5.2.1)$$

where $d\tau$ is the proper time $d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$ (5.2.2)

In the x^α coord., we have

$$0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

multiply by $\frac{\partial x^\lambda}{\partial \xi^\alpha}$, get $\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ (5.2.3a)

where $\Gamma^\lambda_{\mu\nu} \triangleq \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial \xi^\alpha}$ is called the <u>affine connection</u>.

The proper time may also expressed in arb. coord. system x^α:

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu}dx^\nu = -g_{\mu\nu}\,dx^\mu dx^\nu$$

$$g_{\mu\nu} \triangleq \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \qquad (5.2.3b)$$

Note that $g_{\mu\nu}$ as defined by (5.2.3b) transforms under <u>general coord. transf.</u> $x^\alpha \to x'^\alpha$ as:

Under $x^\mu \to x'^\mu$, $g'_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x'^\mu}\frac{\partial \xi^\beta}{\partial x'^\nu}\eta_{\alpha\beta} = \frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\alpha\beta}$

$$= g_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} \quad ✓$$

(<u>Note</u>: For Lorentz, $\frac{\partial x'^\alpha}{\partial x^\beta} = \Lambda^\alpha{}_\beta$.)

We say $g_{\mu\nu}$ is a covariant tensor under general coord transf.

- Transformation of affine Connection:

Claim: under $x^\mu \to x'^\mu$,

$$\Gamma^\lambda_{\mu\nu} \to \Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\Gamma^\rho_{\tau\sigma} + \underbrace{\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}}_{\text{Can also be written as } -\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\sigma}} \quad (5.2.4)$$

$\therefore \Gamma^\lambda_{\mu\nu}$ is not a tensor.

[$\because$ diff wrt. x'^μ: $\frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} = \delta^\lambda_\nu$]

Pf. $\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial \xi^\alpha}{\partial x^\sigma}\right)$

$$= \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial \xi^\alpha}\left[\underbrace{\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial^2 \xi^\alpha}{\partial x^\tau \partial x^\sigma}}_{\text{first term} \checkmark} + \underbrace{\frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}\frac{\partial \xi^\alpha}{\partial x^\sigma}}_{\text{second term} \checkmark}\right]$$

Claim. Define $\left\{ {\lambda \atop \mu\nu} \right\} = \frac{1}{2} g^{\lambda\kappa}\left[\frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\kappa\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right] \quad (5.2.5)$

then $\left\{ {\lambda \atop \mu\nu} \right\}' = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\left\{ {\rho \atop \tau\sigma} \right\} + \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \quad (5.2.6)$

ie same as (5.2.4)!

Pf. Direct Computation.

Therefore if we consider $\Gamma^\lambda_{\mu\nu} - \left\{ {\lambda \atop \mu\nu} \right\}$, then it is a tensor and,

$$\Gamma'^\lambda_{\mu\nu} - \left\{ {\lambda \atop \mu\nu} \right\}' = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\left[\Gamma^\rho_{\tau\sigma} - \left\{ {\rho \atop \tau\sigma} \right\}\right]$$

Now, there always exists a local inertial frame ξ^α in which gravity is absent so in this frame, $\Gamma^\lambda_{\mu\nu} = 0$. Also the first derivatives of $g_{\mu\nu}$ must be zero since there cannot be grav. redshift between infinitesmally separated points.

Therefore, $\Gamma^{\rho}_{\tau\sigma} - \left\{ {\rho \atop \tau\sigma} \right\} = 0$ in the ξ^α frame.

Since it is a tensor, it means it also vanishes in any general frame.

$\therefore \Gamma^{\rho}_{\tau\sigma} = \left\{ {\rho \atop \tau\sigma} \right\}$ in all coordinate system!

$\left\{ {\rho \atop \tau\sigma} \right\}$ is called the christoffel symbol.

NB. 1° (5.2.5) gives the connection in terms of the derivatives of the metric. Note that it is symm. in $\mu \leftrightarrow \nu$.

2° (5.2.3a) is called the geodesic eqn. of motion.

It determines the EOM of a particle moving under the effect of gravity.

non zero $\Gamma^{\lambda}_{\mu\nu}$.

3° According to EEP, physical laws must be form invariant under general coord. transf.

~~we could either~~

To derive the physical laws in the presence of gravity, we could either:

i) write down the physical law in the absence of gravity (ie. SR), and then perform a coord transformation to find the corresponding eqn. in a general frame.

This is what we did above to derive the geodesic eqn. (5.2.3a).

However this method is tedious and not the best in general.

ii) A better method is based on the principle of general covariance. It states that a physical eqn holds in a general grav. field if

α) the field eqn holds in the absense of gravity

ie it agrees with the laws of SR when $g_{\mu\nu} = \eta_{\mu\nu}$ and the affine connection $\Gamma^{\alpha}_{\alpha\beta}$ vanishes.

β) The eqn is general covariant. ie. it preserves its form under a general coord transf $x^\alpha \to x'^\alpha$.

Note that the principle of general covariance is physically equivalent to the EEP.

5.3. Tensors

In order to construct physical equations that are form invariant under general coord. transf., we need to know how the quantities described by the equations behaves under the transformation.

Tensors is a class of objects whose transformation properties are particularly simple.

For instance, a tensor $T^{\mu\lambda}{}_{\nu}$ tranforms under $x \to x'$ as:

$$T'^{\mu\lambda}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} T^{\alpha\beta}{}_{\gamma} \qquad (5.3.1)$$

For example, $g_{\mu\nu}$ is a covariant tensor of 2nd rank.

$\Gamma^{\lambda}_{\mu\nu}$ is not a tensor.

In view of the principle of general covariance, tensors are very useful quantities for building up the physical eqn.

5.4 Covariant derivatives

In general, differentiation of a tensor does not give a tensor.

eg $V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}$

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\alpha}}{\partial x^{\rho}}}_{\text{ok}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha}\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\alpha}}_{\text{destroy the tensor behaviour.}} \qquad (5.4.2)$$

However one can try to construct a covariant derivative so that the second term cancels.

Using (5.2.4), we have

$$\Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \left[\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} - \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} \right] \frac{\partial x'^{\kappa}}{\partial x^{\alpha}} V^{\alpha}$$

↑ precisely cancel last term of (5.4.2)

$\therefore$ $V^{\mu}{}_{;\lambda} \triangleq \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}$ transforms as a tensor (5.4.3)

$$V'^{\mu}{}_{;\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} V^{\alpha}{}_{;\beta} \qquad (5.4.4)$$

$V^{\mu}{}_{;\lambda}$ is called the covariant derivative of V^{μ} wrt x^{λ}.

It could also be denoted as $\nabla_{\lambda} V^{\mu} \triangleq \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}$.

Similarly, the covariant derivative for a covariant vector V_{μ} is defined as $V_{\mu;\lambda} \triangleq \frac{\partial V_{\mu}}{\partial x^{\lambda}} - \Gamma^{\alpha}_{\mu\lambda} V_{\alpha}$ (5.4.5)

and transforms as $V'_{\mu;\lambda} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} V_{\rho;\sigma}$ (5.4.6)

Generally, the covariant derivative of a tensor $T^{\cdots\mu\cdots}_{\cdots\lambda\cdots}$ wrt. x^{ρ} equals

$\frac{\partial T^{\cdots\mu\cdots}_{\cdots\lambda\cdots}}{\partial x^{\rho}}$ plus $\Gamma^{\mu}_{\nu\rho} T^{\cdots\nu\cdots}_{\cdots\lambda}$ for each contravariant index μ

minus $\Gamma^{\kappa}_{\lambda\rho} T^{\cdots\mu\cdots}_{\cdots\kappa\cdots}$ for each covariant index λ. (5.4.7)

Examples: 1. S = a scalar, $S_{;\mu} = \frac{\partial S}{\partial x^\mu}$

2. Covariant curl:

$$V_{\mu;\nu} = \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda$$

symm. in $\mu\nu$

$$\therefore V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu} \qquad (5.4.8)$$

The covariant curl is the ordinary curl.

3. covariant divergence of a contravariant vector:

$$V^\mu{}_{;\mu} \doteq \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} V^\lambda \qquad (5.4.9)$$

now $$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \left\{ \frac{\partial g_{\rho\mu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho} \right\}$$

cancel

$$= \frac{1}{2} g^{\mu\rho} \frac{\partial g_{\rho\mu}}{\partial x^\lambda}$$

4. It is easy to check that $g_{\mu\nu;\tau} = 0$. We say the connection is compatible with the metric.

$$= \frac{1}{2} \frac{\partial}{\partial x^\lambda} \ln g \qquad \delta(\ln \operatorname{Det} M) = \operatorname{Tr}(M^{-1} \delta M)$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\lambda} \sqrt{g}$$

$$\therefore V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g}\, V^\mu) \qquad (5.4.10)$$

An ~~it~~ immediate consequence is that $\int d^4x \sqrt{g}\, V^\mu{}_{;\mu} = 0$

(Covariant form of Gauss's theorem)

one can generalize (5.4.10) to the covariant div. for a tensor:

e.g. For antisymm $A^{\mu\nu} = -A^{\nu\mu}$, it is $A^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g} A^{\mu\nu})$ (Exercise)