6. Manifolds

6-1 Manifolds

Manifolds provide the needed language for the theory of GR.

Informally, a manifold is a space consists of patches that look locally like $\mathbb{R}^n$ and are smoothly sewn together. (n is called the dimension of the manifold)

To make it a mathematically precise definition, we need certain concepts:

- Maps: Given two sets M, N, a map $\phi: \underset{\text{domain}}{M} \to \underset{\text{range}}{N}$ is a relationship that assigns to each element of M exactly one element of N.

- Composition: Given two maps $\phi: A \to B$, $\psi: B \to C$

  the <u>composition</u> $\psi \circ \phi: A \to C$ is defined by the operation

  $$\psi\circ\phi(a) = \psi(\phi(a)) \qquad \forall a \in A.$$

- A map ϕ is <u>one to one</u> if each element of N has at most one element of M mapped into it

- A map ϕ is <u>onto</u> if each element of N has at least one element of M mapped into it.

- A map ϕ that is both one-one + onto is <u>invertible</u>. In this case, we can define the inverse map $\phi^{-1}: N \to M$ by $(\phi^{-1}\circ\phi)(a) = a$

- Consider $\phi : \mathbb{R}^m \to \mathbb{R}^n$

$$(x^1, \cdots, x^m) \longmapsto (y^1, y^2, \cdots, y^n)$$

$$\text{where } y^i = \phi^i(x^1, \cdots, x^m) \qquad i = 1, \cdots, n$$

$\uparrow$ Component function of ϕ

we call a function as C^p if its pth derivative exists and is continuous.

- The map ϕ is C^p if each of its component function is at least C^p.

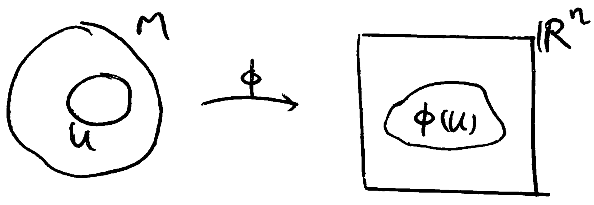
- C^∞ map is continuous and can be differentiated as many times as you like

  It is also called smooth.

- we call two sets M, N diffeomorphic if there exists a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$. The map ϕ is called a diffeomorphism.

- Now we can define manifold

  Let M be a set. A Coord-system of M is given by a subset U of M, together with a one-to-one map $\phi : U \to \mathbb{R}^n$ s.t. $\phi(U)$ is open in $\mathbb{R}^n$.



We say U is an open set in M.

A C^∞- atlas is a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ s.t.

i, U_α cover M i.e. $\bigcup_{\alpha\in I} U_\alpha = M$

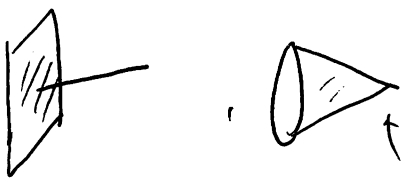
ii, The charts are smoothly sewn together. i.e. For $U_\alpha \cap U_\beta \neq \phi$, two overlapping charts the map $\phi_\alpha \circ \phi_\beta^{-1}$ is C^∞.

Def A C^∞ n-dim manifold is a set M with a maximal atlas

one that contains every possible compatible chart.

e.g. $\mathbb{R}^n$, S^n, T^n, are manifolds

e.g.



not manifold

or manifold with a singularity.

6.2 Riemannian manifold

Metric : A metric is a $\binom{0}{2}$ tensor g. Its components are $g_{\mu\nu}$.

Symmetric

A differentiable manifold with a metric is called a Riemannian manifold.

More precisely,

If g is positive definite i.e. $g(\vec{v},\vec{v}) \geq 0$ for all $\vec{v}$, then it is called Riemannian.

If the metric is indefinite, it is called pseudo-Riemannian.

In this case, an important characterization of the metric is its signature, ~~the number of ...~~ $n_+ - n_-$.

$n_\pm$ = number of positive / negative eigenvalues.

e.g. Lorentz metric η has sig. $= 2$.

Mathematically, one can show that for a Riemannian manifold M. Consider any point P of M, it is always possible to choose a local coord. system such that

$$g_{\alpha\beta}(x^\mu) = \eta_{\alpha\beta} + O((x^\mu)^2) \qquad \text{Here } x^\mu(P) = 0 \qquad (6.2.1)$$

or equivalently :

$$\begin{cases} g_{\alpha\beta}(P) = \eta_{\alpha\beta} & (6.2.2a) \\ \dfrac{\partial}{\partial x^\gamma} g_{\alpha\beta}(P) = 0 & (6.2.2b) \end{cases}$$

This coord. system is called the <u>normal coord system</u>.

physically it just means one can always find a local Lorentz frame such that the effect of gravity are absent.

(6.2.2a)

(6.2.2b)

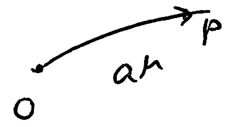
free particles moves on lines that are locally straight in this coord system.

Einstein realized that Riemannian manifold provides precisely the mathematical language to describe the physical situation as required by EEP.

- <u>Riemann normal coord</u>

The basic idea is to use the geodesics through a given point to define the coordinates for nearby points. Let the given point be O and consider some nearby point P. If P & O is close enough, then there exists a unique geodesic joining O to P.

Let a^μ be the components of the unit vector tangent to the geodesics and let s be the geodesic arc length from O to P.



Then the Riemann normal coord. of P is defined by $x^\mu_{(s)} = s\, a^\mu$.

Subsituting into the geodesic eqn

$$0 = \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}$$

One obtain ~~the~~ at the origin O:

$$\Gamma^\mu_{\alpha\beta}\Big|_O = 0 \qquad (6\text{-}2\text{-}3)$$

$$\underbrace{\Gamma^\mu_{\alpha\beta}}_{\text{symmetric}} = \frac{1}{2} g^{\mu\lambda} \left[g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda} \right] = 0 \qquad d \times \frac{d(d+1)}{2} \text{ conditions}$$

$g_{\mu\lambda,\alpha}$ total $\frac{d(d+1)}{2} \times d$ components.

Must be $g_{\mu\nu,\alpha}\Big|_O = 0$ (6-2-4)

Length and volumes :

with a metric, one can measure length of curves and volume of space.

For example, for a metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

and a curve $x^\mu = x^\mu(\lambda)$ $\quad \lambda_0 \le \lambda < \lambda_1$

the length of the curve is given by

$$\ell = \int |ds| = \int_{\lambda_0}^{\lambda_1} \left| g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right|^{1/2} d\lambda \qquad (6.2.5)$$

For volume, it is $V = \int d^4x \sqrt{-g}$ $\quad$ (6.2.6)

in local Lorentz frame, obviously d^4x
In a general coord. x',
$d^4x = d^4x' \underbrace{\det\left(\frac{\partial x}{\partial x'}\right)}_{= \det \Lambda}$
$\Lambda^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}$
Now,
$g_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$
$\therefore \det g = (\det \Lambda)^2$
$\therefore d^4x = d^4x' \sqrt{|\det g|}$

one can show that ds, $d^4x\sqrt{-g}$ are inv. under $x \to x'$.

6.3 Covariant differentiation & Parallel transport

Consider a curve $x^\mu(\tau)$ and a vector field $A^\mu(x)$. A^μ transforms as on the curve,

$$A'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(\tau)$$

If we differentiate $\frac{d}{d\tau}$, then

$$\frac{dA'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dA^\nu}{d\tau}(\tau) + \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}}_{\text{same transformation terms as in second term in (5.2.4).}} \frac{dx^\lambda}{d\tau} A^\nu(\tau)$$

This motivates us to define the covariant derivatives along the curve,

$$\frac{DA^\mu}{D\tau} \triangleq \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau} A^\nu \qquad (6.3.1)$$

then

$$\frac{DA'^\mu}{D\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{DA^\nu}{D\tau} \qquad (6.3.2)$$

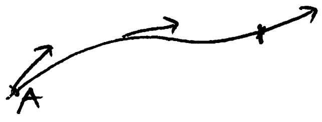
For a covariant vector, the covariant derivatives along a curve is given by

$$\frac{DB_\mu}{D\tau} \triangleq \frac{dB_\mu}{d\tau} - \Gamma^\lambda_{\mu\nu}\frac{dx^\nu}{d\tau} B_\lambda \qquad (6.3.3)$$

This can be extended similarly to general tensor field.

We say a vector (or a tensor) is parallel transported along the curve if

$$\frac{DA^\mu}{D\tau} = 0 \qquad (6.3.4)$$



geodesics : A geodesic is a curve whose tangent vector is parallel transported along the curve.

↑ This is the curved space analogy of the statement that for a straight line in a flat space, its tangent vector at a point is // to the tangent vector at a previous point.

(identical)

[We will show later that a geodesic is a curve which gives the shortest proper distance between points:

i.e. $\delta \int_A^B ds = 0$

]

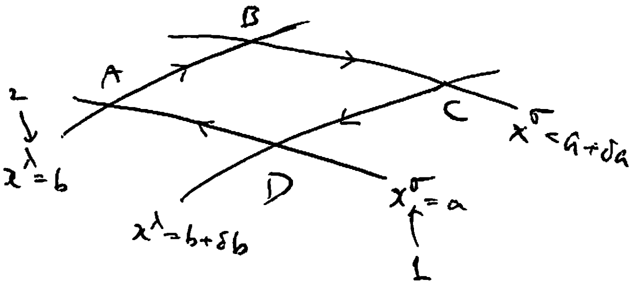
Mathematically,

$$\frac{DU^\mu}{D\tau} = 0 \qquad U^\mu = \frac{dx^\mu}{d\tau} \qquad (6\text{-}3.5)$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \qquad (6\text{-}3\text{-}6)$$

6.4. Curvature tensor.

Consider vector V^μ being ~~pa~~ // transported along the diagram.



$A \to B$: $$V^\alpha(B) = V^\alpha(A) + \int_A^B \frac{\partial V^\alpha}{\partial x^\sigma} dx^\sigma$$

$$= V^\alpha(A) - \int_{x^\lambda = b} \Gamma^\alpha_{\mu\sigma} V^\mu dx^\sigma$$

$B \to C$: $$V^\alpha(C) = V^\alpha(B) - \int_{x^\sigma = a+\delta a} \Gamma^\alpha_{\mu\lambda} V^\mu dx^\lambda$$

$C \to D$: $$V^\alpha(D) = V^\alpha(C) + \int_{x^\lambda = b+\delta b} \Gamma^\alpha_{\mu\sigma} V^\mu dx^\sigma$$

$D \to A$: $$V^\alpha(A)\Big|_{final} = V^\alpha(D) + \int_{x^\sigma = a} \Gamma^\alpha_{\mu\sigma} V^\mu dx^\sigma$$

$$\delta V^\alpha = V^\alpha(A)\Big|_{final} - V^\alpha(A)\Big|_{initial} = -\int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^\sigma}(\Gamma^\alpha_{\mu\lambda} V^\mu) dx^\lambda$$

$$+ \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^\lambda}(\Gamma^\alpha_{\mu\sigma} V^\mu) dx^\sigma$$

$$\simeq \delta a \delta b \left[-\frac{\partial}{\partial x^\sigma}(\Gamma^\alpha_{\mu\lambda} V^\mu) + \frac{\partial}{\partial x^\lambda}(\Gamma^\alpha_{\mu\sigma} V^\mu) \right]$$

eliminate the derivatives of V^α using the (6.3.4), then

$$\delta V^\alpha = \delta a \delta b \underbrace{\left[\Gamma^\alpha_{\mu\sigma,\lambda} - \Gamma^\alpha_{\mu\lambda,\sigma} + \Gamma^\alpha_{\nu\lambda} \Gamma^\nu_{\mu\sigma} - \Gamma^\alpha_{\nu\sigma} \Gamma^\nu_{\mu\lambda} \right]}_{\triangleq R^\alpha_{\mu\lambda\sigma}} V^\mu \qquad (6.4.1)$$

Curvature tensor

In terms of derivatives of g, one can find that

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}\right) \qquad (6\text{-}4.2)$$

Algebraic id:

$$R_{\alpha\beta\mu\nu} = - R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} \qquad (6.4.3)$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \qquad (6\text{-}4\text{-}4)$$

Bianchi id: $R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$

Manifold flat $\Leftrightarrow$ $R^{\alpha}{}_{\beta\mu\nu} = 0 \qquad (6.4.5)$

- Another interpretation of the Riemann tensor is that it gives the commutator of covariant derivatives:

$$[\nabla_\alpha, \nabla_\beta] V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu \qquad (6.4.6\,a)$$

$$[\nabla_\alpha, \nabla_\beta] U_\nu = R_\nu{}^\sigma{}_{\alpha\beta} U_\sigma \qquad (6.4.6\,b)$$

Ricci tensor: $R_{\alpha\beta} \triangleq R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha}$ (6.4.7)

Ricci scalar: $R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}$ (6.4.8)

Einstein tensor: $G_{\alpha\beta} \triangleq R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = G_{\beta\alpha}$ (6.4.9)

It has the important property: $G_{\alpha\beta;}{}^{\beta} = 0$ divergence free! (6.4.10)

pf. Bianchi id. $\Rightarrow$ $g^{\alpha\mu}\left[R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}\right] = 0$

$g_{\alpha\beta;\mu}=0$ $\Leftrightarrow$ $R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0$

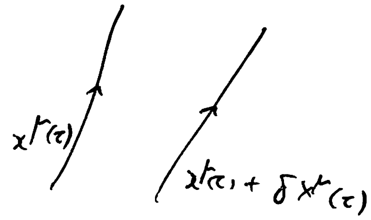
$g^{\beta\nu}\left[\qquad\right] = 0$

$\Rightarrow R_{;\lambda} - R^{\mu}{}_{\lambda;\mu} - R^{\mu}{}_{\lambda;\mu} = 0$

i.e. $\left(2R^{\mu}{}_{\lambda} - \delta^{\mu}{}_{\lambda} R\right)_{;\mu} = 0$ //

- <u>geodesic deviations</u>

Consider two nearby free falling particle in a gravitational field:



EOM are

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\, \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\frac{d^2(x^\mu + \delta x^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x+\delta x)\, \frac{d(x^\nu + \delta x^\nu)}{d\tau}\, \frac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0$$

Subtracting and use $\Gamma^\mu_{\nu\lambda}(x+\delta x) = \Gamma^\mu_{\nu\lambda}(x) + \Gamma^\mu_{\nu\lambda,\rho}\,\delta x^\rho$

Keep to 1st order, we have

$$\frac{d^2 \delta x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda,\rho}\,\delta x^\rho\, \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau} = 0$$

This can be rewritten as: (Exercise)

$$\frac{D^2}{D\tau^2}\delta x^\mu = +R^\mu{}_{\alpha\beta\nu}\,\delta x^\nu\, \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

In flat space, $R=0$ and so $\dfrac{D^2 \delta x^\lambda}{D\tau^2} = 0$

In curved space, $\dfrac{D^2 \delta x^\lambda}{D\tau^2} \neq 0$ and represents relative acceleration between free falling particles!

Pf: $\frac{D}{D\tau}\delta x^\mu = \frac{d\delta x^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\delta x^\nu$

$$\frac{D^2}{D\tau^2}\delta x^\mu = \frac{d}{d\tau}\left(\frac{D\delta x^\mu}{d\tau}\right) + \Gamma^\mu_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{D\delta x^\alpha}{D\tau}$$

$$= \frac{d^2}{d\tau^2}\delta x^\mu + \frac{d}{d\tau}\left(\Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\delta x^\nu\right) + \Gamma^\mu_{\alpha\beta}\frac{dx^\beta}{d\tau}\left(\frac{d\delta x^\alpha}{d\tau} + \Gamma^\alpha_{\nu\lambda}\frac{dx^\lambda}{d\tau}\delta x^\nu\right)$$

$$= \frac{d^2}{d\tau^2}\delta x^\mu + \Gamma^\mu_{\nu\lambda,\rho}\frac{dx^\rho}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu + \Gamma^\mu_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{d\delta x^\alpha}{d\tau}$$

$$+ \Gamma^\mu_{\nu\lambda}\frac{d^2x^\lambda}{d\tau^2}\delta x^\nu + \Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\nu\lambda}\frac{dx^\beta}{d\tau}\frac{dx^\lambda}{d\tau}\delta x^\nu$$

$$+ \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\tau}\frac{d\delta x^\nu}{d\tau} \qquad -\Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta x^\nu$$

$$\bigcirc = -\Gamma^\mu_{\nu\lambda,\rho}\delta x^\rho\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$\therefore \frac{D^2}{D\tau^2}\delta x^\mu = \delta x^\rho\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\Bigg[\underbrace{\underset{\substack{\parallel \\ \Gamma^\mu_{\alpha\rho,\beta}}}{\Gamma^\mu_{\rho\alpha,\beta}} - \Gamma^\mu_{\alpha\beta,\rho} + \Gamma^\mu_{\gamma\alpha}\Gamma^\gamma_{\rho\beta}}_{+R^\mu_{\alpha\beta\rho}} - \overset{\substack{\Gamma^\mu_{\gamma\beta}\Gamma^\gamma_{\alpha\rho} \\ \parallel}}{\Gamma^\mu_{\gamma\rho}\Gamma^\gamma_{\alpha\beta}}\Bigg]$$

$$= R^\mu_{\alpha\beta\rho}\,\delta x^\rho\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

//