7. Physics in curved spacetime

7.1 Principle of Covariance

We have understood that :

Spacetime is described by a curved manifold $(M, g_{\mu\nu})$ in the presence of gravity.

Need to do two things: 1° understand how spacetime is curved by gravity

This is governed by the Einstein eqn.

(will do it later)

2° Understand how physics look like in a curved spacetime.

obj. 2° is simpler. We can use the principle of covariance as discussed before (P.5.7)

An useful rule is "replace comma by semi-colon".

Consider for example, the conservation law of particles in SR:

$$(nU^\alpha)_{,\alpha} = 0 \qquad (7.1.1)$$

A possible covariant form that reduces to the law (7.1.1) in the absense of gravity

is

$$(nU^\alpha)_{;\alpha} = 0 \qquad (7.1.2)$$

However, $(nU^\alpha)_{;\alpha} = qR$ is also possible! (7.1.3)

which is true depends on the experiment.

At low energies (large scale), we can ignore effect of curvature $\left(\sum_n (R/L)^n\right)$,

and so (7.1.2) can be thought of as the low energy leading approx.

We will take this point of view below (ie curvature correction will be ignored. gravity effects will be taken into account by the connection Γ)

7.2. Particle mechanics

Consider a particle in SR, with 4-vel. U^α & Spin S_α, then

$$\begin{cases} \dfrac{dU^\alpha}{d\tau} = 0 & U^\alpha = \dfrac{d\xi^\alpha}{d\tau} \quad (7.2.1) \\ \dfrac{dS_\alpha}{d\tau} = 0 & (7.2.2) \end{cases}$$

Moreover $S^\alpha U_\alpha = 0$ —(7.2.3) since in the rest frame, $S_\alpha = (0, \vec{S})$, $U_\alpha = (1, 0)$

To build the physical law in curved spacetime.

Step 1° Define the quantity in general coord system by:

$$\begin{cases} U^\mu = \dfrac{\partial x^\mu}{\partial \xi^\alpha} U_f^\alpha = \dfrac{dx^\mu}{d\tau} \\ S_\mu = \dfrac{\partial \xi^\alpha}{\partial x^\mu} S_{f\alpha} \end{cases}$$

U_f^α, $S_{f\alpha}$ are the components of U & S in the free falling coord. system.

Step 2° Note that $\dfrac{DU^\mu}{D\tau}$, $\dfrac{DS_\mu}{D\tau}$ are tensors and that they reduces to $\dfrac{dU^\mu}{ds}$, $\dfrac{dS_\mu}{d\tau}$ in the free falling coord.

Therefore by the principle of Covariance, the physical eqn. generalizing (7.2.1) & (7.2.2) are

$$\frac{DU^\mu}{D\tau} = 0 \quad (7.2.4)$$

$$\frac{DS_\mu}{D\tau} = 0 \quad (7.2.5)$$

and

$$S^\mu U_\mu = 0 \quad (7.2.6)$$

The eqn (7.2.4) & (7.2.5) simply states that U^μ & S^μ are parallel transported along the geodesics.

We note that $S^\mu S_\mu$ is an invariant:

$$\frac{d}{d\tau}(S_\mu S^\mu) = \frac{D}{D\tau}(S_\mu S^\mu) = 0 \qquad (7.2.7)$$

If the particle experience a non-grav. force f^μ, then

$$\frac{Du^\mu}{D\tau} = \frac{f^\mu}{m} \qquad (7.2.8)$$

$$\text{or } m\frac{d^2x^\mu}{d\tau^2} = f^\mu \underbrace{- m\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}}_{\text{gravitational force !}}$$

Thomas precession

In SR, an object (eg. particle) can precess even in the absense of an external torque. This is called the Thomas precession.
$\left(\frac{d\vec{L}}{dt} = \vec{\tau}\right)$

To see this, consider a particle in the presence of force f^μ, but not torque.

In the instaneous rest frame of the particle, $\frac{d\vec{x}}{dt} = 0$ and $\frac{d\vec{S}}{dt} = 0$

In an arb. inertial frame, this can be expressed as $\frac{dS^\alpha}{d\tau} = \Phi u^\alpha$ for some scalar Φ.

To get Φ, we note that

$$S^\alpha u_\alpha = 0$$

$$\Rightarrow 0 = \frac{dS^\alpha}{d\tau}u_\alpha + S^\alpha\frac{du_\alpha}{d\tau} = \Phi \underbrace{u_\alpha u^\alpha}_{-1} + S^\alpha\frac{f_\alpha}{m}$$

$$\Rightarrow \Phi = \frac{S_\alpha f^\alpha}{m}$$

Hence,
$$\frac{dS^\alpha}{d\tau} = \left(S_\beta\frac{f^\beta}{m}\right)u^\alpha \qquad (7.2.9)$$

This is called Thomas precession (1926).

In a general gravity field, we have

$$\frac{DS^\mu}{D\tau} = (S_\nu \frac{f^\nu}{m}) u^\mu = S_\nu \frac{Du^\nu}{D\tau} u^\mu \qquad (7.2.10)$$

We say a spin S^μ is Fermi transport if it satisfies the eqn (7.2.10)

7.3 Electrodynamics

Maxwell eqn. can be written as

$$\begin{cases} \partial_\alpha F^{\alpha\beta} = -J^\beta & (7.3.1) \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 & (7.3.2) \end{cases}$$

To find the eqn. in gravity field, we follows the steps above:

Step. 1°. Define for general frame

$$F^{\mu\nu} = \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} \tilde{F}^{\alpha\beta} \qquad (7.3.3)$$

$$J^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha} \tilde{J}^\alpha$$

in local inertial frame.

Step. 2° propose the Covariant form:

$$\begin{cases} F^{\mu\nu}{}_{;\mu} = -J^\nu & (7.3.4) \\ F_{\mu\nu;\lambda} + \cdots = 0 & (7.3.5) \end{cases}$$

By principle of general covariance, they should be correct in general grav. field.

7.4 Physics in Slightly Curved Spacetime

The above considerations have been general and hold for arb. grav. field.

The difficult question is how gravity field (or $g_{\mu\nu}$) is affected by other objects (ie. question 1.) . We will leave it in the next chapter.

For now, lets consider a weak field to get further understandings of the effects of gravity on physics (non-grav.).

First, weak field $\leftrightarrow$ slightly curved spacetime.

ie.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \text{Small corrections}$$

Consider a case where space-isotropy is preserved. Then

$$ds^2 = -(1+2\phi)dt^2 + (1+2B)(dx^2+dy^2+dz^2) \tag{7.4.1}$$

where $|\phi|, |B| \ll 1$.

Turns out for a source of mass M, we have $-B = \phi = -\frac{GM}{r}$.

Look at geodesic eqn. $\nabla_u u^\mu = 0$ $\qquad u^\mu = \frac{dx^\mu}{d\tau}$

or in terms of mom. $p^\mu = m u^\mu$, then

$$\nabla_p p^\mu = 0$$

or
$$m\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\, p^\alpha p^\beta = 0 \tag{7.4.2}$$

Consider a nonrelativistic particle, $p^0 \gg p^i$,

then
$$m\frac{dp^\mu}{d\tau} + \Gamma^\mu_{00}(p^0)^2 = 0 \tag{7.4.3}$$

$\mu=0$: $\quad m\dfrac{dp^0}{d\tau} + \Gamma^0_{00}(p^0)^2 = 0$

diagonal,

now $\Gamma^0_{00} = \frac{1}{2} g^{0\alpha}\left(2g_{\alpha 0,0} - g_{00,\alpha}\right)$

$= \frac{1}{2} g^{00}\left(g_{00,0}\right)$

$= \frac{1}{2}\dfrac{1}{-(1+2\phi)} \cdot (-2\phi)_{,0}$

$= \phi_{,0} + O(\phi^2)$

replace $(p^0) = m$ on the LHS.

$\therefore \quad \dfrac{dp^0}{d\tau} = -m\dfrac{\partial\phi}{\partial\tau} \qquad$ ie. $\dfrac{d}{d\tau}(p^0 + m\phi) = 0$

This is the statement of conservation of energy.

ϕ = grav. potential $\quad , \quad m\phi = -\dfrac{GmM}{r}$.

$\mu = i, \quad m\dfrac{dp^i}{d\tau} + \Gamma^i_{00}(p_0)^2 = 0$

$\Rightarrow \quad \dfrac{dp^i}{d\tau} = -m\,\Gamma^i_{00}$

now $\Gamma^i_{00} = \frac{1}{2} g^{i\alpha}\left(2g_{\alpha 0,0} - g_{00,\alpha}\right) \qquad\qquad g^{i\alpha} = \delta^{i\alpha}(1-2\phi)^{-1}$

$= \frac{1}{2}(1-2\phi)^{-1}\delta^{ij}\left(2g_{j0,0} - g_{00,j}\right)$

$= +\frac{1}{2}(1-2\phi)^{-1}(2\phi)_{,j}\,\delta^{ij}$

$= \phi_{,j}\,\delta^{ij} + O(\phi^2)$

$\therefore \dfrac{dp^i}{d\tau} = -m\,\phi_{,i}$

ie. $\dfrac{d\vec{p}}{d\tau} = -m\nabla\phi \qquad$ Newton's law of gravitation!

We therefore reproduce the standard results of Newtonian theory in the weak field approximation. This is of course necessary since we have had plenty of experimental verification of the Newtonian theory in weak field.

7.4 Conserved quantities

Consider the ~~geodesic eqn~~ particle motion again and note that

$$p^\alpha p_{\beta;\alpha} = 0 \qquad \text{(parallel transport)}$$

$$\Rightarrow \quad p^\alpha p_{\beta,\alpha} - \Gamma^\gamma{}_{\beta\alpha} p^\alpha p_\gamma = 0$$

$$\Rightarrow \quad m \frac{dp_\beta}{d\tau} = \Gamma^\gamma{}_{\beta\alpha} p^\alpha p_\gamma$$

Now $\Gamma^\gamma{}_{\alpha\beta} p^\alpha p_\gamma = \frac{1}{2} g^{\gamma\nu} (g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) p^\alpha p_\gamma$

$$= \frac{1}{2} (\qquad \text{''} \qquad) p^\nu p^\alpha$$

Symmetric

$$= \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

$$\therefore \quad m \frac{dp_\mu}{d\tau} = \frac{1}{2} g_{\alpha\beta,\mu} p^\alpha p^\beta \qquad (7.4.1)$$

$\therefore$ component p_μ is conserved if $g_{\alpha\beta}$ is independent of x^μ for some fixed μ.

eg. if metric is stationary, then $g_{\alpha\beta,0} = 0$ and so p_0 is conserved!

$-p_0$ is usually called the energy of the particle.

The frame in which the metric is stationary is special and is usually called the "lab frame". To see more explicitly the meaning of p_0, let us consider

$$p_\mu p^\mu = -m^2 = -(1+2\phi)(p^0)^2 + (1-2\phi)\vec{p}^2$$

$$\Rightarrow \quad (p^0)^2 = (m^2 + (1-2\phi)\vec{p}^2)/(1+2\phi) \qquad (7.4.2)$$

For, $|\phi| \ll 1$, $m \gg |\vec{p}|$,

$$(p^0)^2 \simeq m^2\left(1 - 2\phi + \vec{p}^{\,2}/m^2\right)$$

or $$p^0 \simeq m\left(1 - \phi + \frac{\vec{p}^{\,2}}{2m^2}\right)$$

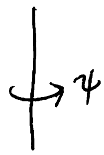
$$p_0 \cong g_{00}\, p^0 = -(1+2\phi)\, p^0$$

$$\Rightarrow -p_0 \simeq m\left(1 + \phi + \frac{\vec{p}^{\,2}}{2m^2}\right) = m + m\phi + \frac{\vec{p}^{\,2}}{2m} \qquad (7.4.3)$$

rest mass — Newtonian gravitional energy & KE.

In general, a general grav field will not be stationary in any frame, so no conserved energy can be defined!

- Similarly, if a metric is axially symmetric, then coord can be found s.t. $g_{\alpha\beta}$ is indep of the angle ψ.



p_ψ will be conserved.

In non-rel. limit, $p_\psi \simeq g_{\psi\psi}\, p^\psi \simeq g_{\psi\psi}\, m \frac{d\psi}{dt} = m\, g_{\psi\psi}\, \Omega$

angular velocity.

For nearly flat metric, $g_{\psi\psi} \simeq r^2$

and $p_\psi \simeq m r^2 \Omega$ = Newtonian def. of angular momentum.

- More about conservation laws for a system of particles (including self-gravitation) later.