8. Einstein eqn.

8.1 Derivation

Let us first recall the Newtonian limit of a particle moving in a weak gravitational field. Assume nonrelativistic motion so that can ignore $\frac{dx^i}{d\tau}$

EOM
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \left|\frac{dt}{d\tau}\right| \gg \left|\frac{dx^i}{d\tau}\right| \tag{8.1.1}$$

For stationary field,
$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu}$$

For weak field,
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

To leading order in h,
$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^\nu}$$

$$\therefore \begin{cases} \dfrac{d^2\vec{x}}{d\tau^2} = \dfrac{1}{2}\left(\dfrac{dt}{d\tau}\right)^2 \nabla h_{00} \\[2ex] \dfrac{d^2 t}{d\tau^2} = 0 \end{cases} \tag{8.1.2}$$

$$\Rightarrow \quad \frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\nabla h_{00} \tag{8.1.3}$$

Compared with Newtonian result: $\frac{d^2\vec{x}}{dt^2} = -\nabla\phi$, $\phi = -\frac{GM}{r}$

$$\Rightarrow h_{00} = 2\phi \times (-1) + \text{const.} \tag{8.1.4}$$

Let $\phi = 0$ at spatial infinity and since h_{00} should be zero at infinity, $\therefore$ const. $= 0$
(Minkowski at infinity)

$$\therefore \boxed{g_{00} = -(1+2\phi)} \tag{8.1.5}$$

In general ϕ is a soln to the Poisson eqn.

$$\nabla^2\phi = 4\pi G\rho \tag{8.1.6}$$

and ρ is a component of the energy-momentum tensor.

$$T_{00} \simeq \rho$$

$$\therefore \nabla^2 g_{00} = -8\pi G\, T_{00} \qquad (8.1.7)$$

This eqn holds only for non-relativistic matter in weak field. It is not even Lorentz covariant.
But it leads us to guess that the general field eqn takes the form:

$$G_{\alpha\beta} = -8\pi G\, T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is a differential operator, 2nd rank tensor and $G^{\alpha\beta}{}_{;\beta} = 0$ due to energy-mom. conservation.

Q. What is the form of $G_{\alpha\beta}$?

Look for a $\binom{0}{2}$ Tensor which is a combination of $g_{\mu\nu,\lambda\rho}$, $g_{\mu\nu,\lambda}$, $g_{\mu\nu}$.

Try $G_{\alpha\beta} = R_{\alpha\beta} + \mu g_{\alpha\beta} R + \Lambda g_{\alpha\beta}$

$G_{\alpha\beta;}{}^{\beta} = 0$ if $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta}$

then $G_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = k T_{\alpha\beta}$, k = const. $\qquad (8.1.8)$

<u>remark</u>: 1° k is a constant to be determined when compared to experiments

Turns out need $k = \dfrac{8\pi G}{c^2}$

2° Alternatives to Einstein eqn. is possible. But so far Einstein eqn. is the simplest/aesthetic and it agrees well with all experiments.

3° Einstein eqn is classical. A complete Quantum theory of gravity is ~~still~~ open (but string is the best candidate)

Geometrized unit

Constant	SI	Geometrized unit
c	$2.998 \times 10^8\ ms^{-1}$	1
G	$6.674 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$	1

Thus in geometrized unit, $1s = 2.998 \times 10^8\ m$

$$1kg = 7.425 \times 10^{-28}\ m \qquad G/c^2 = 1 = 7.425 \times 10^{-28}\ \frac{m}{kg}$$

both measured in terms of meters.

8.2 Einstein eqn. (Comments)

1° Λ is called the cosmological constant. It represents a replusion.

It corresponds to a vacuum energy momentum tensor of

$$T_{\alpha\beta} = -\frac{\Lambda}{8\pi G}\, g_{\alpha\beta}\,.$$

Historically, Einstein initially did not consider this term.

He introduced $\Lambda \neq 0$ later to obtain a static cosmological soln.

But it was later observed that the universe expands and the Λ-term is not ~~has~~ needed.

Einstein refers to it as his biggest mistake in life.

But ~~later~~ recent astronomical observations suggest that Λ is tiny and nonzero.

2° Einstein eqn is a set of 10 eqns ($G_{\alpha\beta}$ symmetric)

with four constraints $G^{\alpha\beta}{}_{;\beta} = 0$

on the metric $g_{\mu\nu}$.

The metric $g_{\mu\nu}$ has 10 components but in fact only 6 of them are indep.

Since $g_{\mu\nu}$ change in general under a coord. transf. and there are four

indep. coordinate change:

$$x^\mu \to x^\mu + \delta x^\mu$$

↖ 4 arb. functions.

So in principle can solve Einstein eqn to obtain $g_{\mu\nu}$.

3° Einstein eqn is nonlinear PDE. Very difficult to solve in general.

Numerical soln is often needed.

4° Given a metric, there are useful analytical calculational tools for

computing the curvatures $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$, R etc.

eg. Mathematica, maple, maxima

↖ free.

8.3 Weak gravitational fields

By definition, a weak gravity field is one in which spacetime is nearly flat.

i.e. there exists coordinates such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 \qquad (8.3.1)$$

Note that, even for a flat metric, there exists coord. such that

g_{ij} could be big.

$$\text{e.g. } \mathbb{R}^2 \text{ in spherical coord. } \quad g_{rr} = 1 \ , \ g_{\theta\theta} = r^2$$

$$\text{Cartesian coord.} \ , \ g_{xx} = 1 = g_{yy} \qquad (8.3.2)$$

Although other coordinates choices are possible,
it is important to choose the nearly Lorentz coordinates.

$\left.\begin{array}{l}\text{Physics is most clear} \\ \text{Calculation is easier}\end{array}\right\}$ in this coord system

Two interesting features arise with the nearly Lorentz coordinates:

1° Background Lorentz transf.
Under a Lorentz transf. $x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ $\quad , \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$

$$h_{\alpha\beta} \text{ transform as } \tilde{h}_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta h_{\mu\nu} \qquad (8.3.2)$$

$$\text{Pf. } g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \underbrace{g_{\alpha\beta}}_{\eta_{\alpha\beta} + h_{\alpha\beta}}$$

$$= \underbrace{\Lambda\Lambda\eta}_{\eta} + \Lambda\Lambda h$$

$$\overset{!}{=} \eta + \tilde{h}$$

//

2° Gauge transf.

Under an infinitesimal coord change.

infintesimal

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x^\alpha) \qquad , \| \xi^\alpha{}_{,\beta} \| \ll 1$$

then we have $\dfrac{\partial \tilde{x}^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta + \xi^\alpha{}_{,\beta}$

$$\frac{\partial x^\alpha}{\partial \tilde{x}^\beta} = \delta^\alpha{}_\beta - \xi^\alpha{}_{,\beta} + o(|\xi^\alpha{}_{,\beta}|^2)$$

hence $\tilde{g}_{\alpha\beta} = \dfrac{\partial x^\mu}{\partial \tilde{x}^\alpha} \dfrac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}$

$$= \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + o(\xi^2)$$

Hence to $h_{\alpha\beta}$ transforms as

$$h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \qquad (8.3.3)$$

This is called a gauge transf. for $h_{\mu\nu}$.

cf. $A_\mu \to \partial_\mu \Lambda + A_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ inv.

$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \qquad H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$ inv.

<u>Riemannian tensor:</u>

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})$$

Sub. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}) \qquad (8.3.4)$$

it is easy to see that $R_{\alpha\beta\mu\nu}$ is inv. under (8.3.3).

$$\delta R_{\alpha\beta\mu\nu} = \frac{1}{2}\begin{pmatrix} \cancel{\xi_{\alpha,\nu\beta\mu}} + \cancel{\xi_{\nu,\alpha\beta\mu}} - \cancel{\xi_{\alpha,\mu\beta\nu}} - \cancel{\xi_{\mu,\alpha\beta\nu}} \\ + \cancel{\xi_{\beta,\mu\alpha\nu}} + \cancel{\xi_{\mu,\beta\alpha\nu}} - \cancel{\xi_{\beta,\nu\alpha\mu}} - \cancel{\xi_{\nu,\beta\alpha\mu}} \end{pmatrix} = 0$$

Einstein eqn in weak field

By weak field, it means we keep only first order terms in h.

As a result, indices of h are raised & lowered by η:

$$h^{\mu}{}_{\beta} = g^{\mu\alpha} h_{\alpha\beta} = \eta^{\mu\alpha} h_{\alpha\beta} + o(h^2)$$

$$h^{\mu\nu} = \eta^{\nu\beta} h^{\mu}{}_{\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}$$

$$h \hat{=} h^{\alpha}{}_{\alpha}$$

Define the trace reverse of $h_{\alpha\beta}$:

$$\bar{h}^{\alpha\beta} \triangleq h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h \qquad (8.3.3)$$

$$\text{then} \quad \bar{h} \hat{=} \bar{h}^{\alpha}{}_{\alpha} = -h \qquad (8.3.4)$$

Moreover, $$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h} \qquad (8.3.5)$$

$$\therefore \quad G_{\alpha\beta} = -\frac{1}{2}\left[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu} + o(h^2)\right] \qquad (8.3.6)$$

then the Einstein eqn will be greatly simplified if we choose the gauge

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \qquad (8.3.7)$$

In this case, $$G_{\alpha\beta} = -\frac{1}{2}\bar{h}_{\alpha\beta,\mu}{}^{,\mu} = -\frac{1}{2}\Box\bar{h}_{\alpha\beta}. \qquad (8.3.8)$$

we can take this gauge since generally, under a coord transf. $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$

$$\bar{h}_{\mu\nu} \to \tilde{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\, \xi^{\alpha}{}_{,\alpha}$$

$$\therefore\ \tilde{\bar{h}}^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}{}_{,\nu}$$

$\therefore$ by choosing ξ^μ s.t. $\Box \xi^\mu = \bar{h}^{\mu\nu}{}_{,\nu}$ (8.3.9)

← always possible

we can take the gauge (8.3.8).

This is called the <u>Lorentz gauge</u>. (cf EM: $\partial^\mu A_\mu = 0$)

Note that Lorentz gauge is not unique,

since $\xi^\mu + \eta^\mu$ still satisfy (8.3.9) so long as $\Box \eta^\mu = 0$

$\therefore$ Lorentz gauge is really a class of gauge.

Weak-field Einstein eqn is

$$\Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu} \qquad (8.3.10)$$

These are called the field eqn in the <u>linearized theory</u>.

Newtonian limit : $|\phi| \ll 1$ ← weak gravity
is
characterized by $|v| \ll 1$ ← low speed

Generically, Small velocities implies that

$$|T^{00}| \gg |T^{0i}| \gg |T^{ij}| \qquad (8.4.1)$$

But there could be special case where the inequalities ~~does~~ do not hold.

eg for a spherical star, $T^{0i} = 0$ identically.

But ~~th~~ (8.4.1) holds generically.

~~Also~~ Again, generically, (8.4.1) translates to

$$|\bar{h}^{00}| \gg |\bar{h}^{0i}| \gg |\bar{h}^{ij}| \qquad (8.4.2)$$

Note that $\Box \bar{h}^{\alpha\beta} \sim T^{\alpha\beta}$

we can generally add a homogeneous soln $\bar{h}^{\alpha\beta}_{(homo)}$ to $\bar{h}^{\alpha\beta}$ and so (8.4.2)

would not hold.

The homogeous sol. $\Box \bar{h}^{\alpha\beta} = 0$ represents gravity wave. (8.4.3)

of course, there is no gravity wave in Newtonian theory. Therefore ~~we~~ if we ~~take~~ take (8.4.2), then we will reproduce the Newtonian limit (without gravity wave!.)!

As a result, we can expect the domaniant eqn to be in the Newtonian limit

$$\Box \bar{h}^{00} = -16\pi\rho \qquad (8.4.4)$$

where we have used $T^{00} = \rho + O(\rho v^2)$

Also, for source of velocity $\vec{v}$, $\frac{\partial}{\partial t} \simeq v \frac{\partial}{\partial x}$ and so

$$\Box = \nabla^2 + O(v^2 \nabla^2)$$

$$\therefore \nabla^2 \bar{h}^{00} = -16\pi \rho$$

Compared to $\nabla^2 \phi = 4\pi\rho$

we have $\bar{h}^{00} = -4\phi$

$h = -\bar{h}^{\alpha}{}_{\alpha} = \bar{h}^{00}$

2) $h^{00} = -2\phi$,

$h^{xx} = h^{yy} = h^{zz} = -2\phi$

$$ds^2 = -(1+2\phi)\,dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$$

This is the form of the metric we used before to reproduce the Newton's eqn of motion in a gravity field.

Thus this justify i. Einstein eqn in the weak field limit reproduces Newtonian gravity.

ii. The choice of $k = 8\pi$

Comments: i, Corrections to Newtonian gravity are called Post-Newtonian effects.

eg 1. precession of the perihelion of Mercury
2. bending of lights by the sun.
} Solar

3. Shrinking of the orbit of the binary Pulsar ← outside solar System (test gravitational wave)

Far field of stationary relativistic sources

The metric (8.4.5) was obtained under the limit $\begin{cases} |\phi| \ll 1 & \text{weak gravity} \\ |v| \ll 1 & \text{low speed.} \end{cases}$

We now like to consider another limit where we could solve the weak field Einstein eqn (8.3.10).

Consider a source which is localized in a certain compact region of space. Then far away from the source, gravity is weak and we can treat the weak field Einstein eqn. (8.3.10). Also $T_{\mu\nu} = 0$ away from the source. Therefore,

$$\Box \bar{h}^{\mu\nu} = 0 \qquad (8.4.6)$$

(Note that we did not assume field is weak everywhere as in the derivation of (8.4.5). We simply find a region, the far region, where the gravity is weak.)

To simplify further (8.4.6), we consider a stationary source (independent of time) and so $\bar{h}^{\mu\nu}$ should be time indep. also. Then

$$\nabla^2 \bar{h}^{\mu\nu} = 0 \qquad (8.4.7)$$

This has soln.

$$\bar{h}^{\mu\nu} = \frac{A^{\mu\nu}}{r} + O\left(\frac{1}{r^2}\right) \leftarrow \text{spherical harmonics}$$

gauge condition $\bar{h}^{\mu\nu}{}_{,\nu} = 0 = \bar{h}^{\mu j}{}_{,j} = -A^{\mu j}\frac{x_j}{r^2} + O(\frac{1}{r^3})$

$\Rightarrow A^{\mu j} = 0 \qquad \forall \mu, j$

Only A^{00} survive in leading order in $\frac{1}{r}$.

ie far from the source, $|\bar{h}^{00}| \gg |\bar{h}^{ij}|$, $|\bar{h}^{00}| \gg |\bar{h}^{0j}|$.

we may define the "Newtonian potential" at far region to be

$$(\phi)_{\text{relativistic far field}} = -\frac{1}{4}(\bar{h}^{00})_{\text{far field}} \qquad (8.4.8)$$

Far from the ~~source~~ $\boxed{\text{Newtonian}}$, $(\phi)_{\text{Newton far field}} = -\frac{M}{r} + O(1/r^2)$

we may ~~[illegible]~~ take the same definition of mass for a relativistic source,

$$(\phi)_{\text{rel. far field}} = -\frac{M}{r} + O(\frac{1}{r^2}) \qquad (8.4.9)$$

NB The mass defined this way is a measure of grav. field at infinity. it is not a "counting".

then need to choose $A^{00} = 4M$

$\bar{h} = \bar{h}^{\alpha}{}_{\alpha} = -\bar{h}^{00} = -\frac{4M}{r}$

$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h} \Rightarrow h^{00} = \bar{h}^{00} + \frac{1}{2}(-\bar{h}^{00}) = \frac{1}{2}\bar{h}^{00} = +\frac{2M}{r} = -2\phi \quad (\phi \triangleq -\frac{M}{r})$

$= -h_{00}$

$h^{xx} = \bar{h}^{xx} - \frac{1}{2}\eta^{xx}\bar{h} = -\frac{1}{2}\bar{h} = \frac{1}{2}\bar{h}^{00}$ etc.

$= +h_{xx}$

$$\therefore ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2) \qquad (8.4.10)$$

Energy, momentum and angular momentum for a finite gravitational source.

At far distance, we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{so that } h_{\mu\nu} \to 0 \quad \text{at } r \to \infty .$$

The linear part of $R_{\mu\nu}$ is

$$R^{(1)}_{\mu\kappa} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h^\lambda{}_\kappa}{\partial x^\lambda \partial x^\mu} + \Box h_{\mu\kappa} \right) \qquad (8.4.11)$$

The exact Einstein eqn reads:

$$R^{(1)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa} R^{(1)\lambda}{}_\lambda = -8\pi G \left(T_{\mu\kappa} + t_{\mu\kappa} \right) \qquad (8.4.12)$$

$$t_{\mu\kappa} \triangleq \frac{1}{8\pi G}\left(R_{\mu\kappa} - R^{(1)}_{\mu\kappa} - \frac{1}{2} g_{\mu\kappa} R^\lambda{}_\lambda + \frac{1}{2}\eta_{\mu\kappa} R^{(1)\lambda}{}_\lambda \right) \qquad (8.4.13)$$

we could interpret (8.4.12) as saying $t_{\mu\kappa}$ ($T_{\mu\kappa}$) is the energy mom. tensor of gravitational field (matter)

And $\tau_{\mu\kappa} = T_{\mu\kappa} + t_{\mu\kappa}$ as the total energy momentum tensor of the system.

$$\tau^{\mu\kappa} \triangleq \eta^{\mu\nu}\eta^{\kappa\lambda}\left(T_{\nu\lambda} + t_{\nu\lambda} \right) \qquad (8.4.14)$$

Support of this identification:

1) The LHS of (8-4-12) satisfies $\frac{\partial}{\partial x^\nu}\left(R^{(1)\nu\lambda} - \frac{1}{2}\eta^{\nu\lambda} R^{(1)\mu}{}_{\mu}\right) = 0$

Hence $\tau^{\nu\lambda}$ is locally conserved: $\tau^{\nu\lambda}{}_{,\lambda} = 0$ (8-4-15)

We may interpret $P^\lambda = \int d^3x\, \tau^{0\lambda}$ as the total energy-mom. vector of the system.

One may show that P^0 is always positive and equal zero only for a free space (positive energy theorem!)

2) $\tau^{\nu\lambda}$ is symm., hence

$$\partial_\mu M^{\mu\nu\lambda} = 0 \qquad \text{for } M^{\mu\nu\lambda} \triangleq \tau^{\mu\lambda}x^\nu - \tau^{\mu\nu}x^\lambda$$

We can thus interpret $M^{0\nu\lambda}$ & $M^{i\nu\lambda}$ as the density & flux of the total angular momentum of the system.

3) If we compute $t_{\mu\kappa}$ as a power series in h, then

$$t_{\mu\kappa} = \frac{1}{8\pi G}\Big[\underbrace{-\frac{1}{2}h_{\mu\kappa}R^{(1)\lambda}{}_{\lambda} + \frac{1}{2}\eta_{\mu\kappa}h^{\rho\sigma}R^{(1)}_{\rho\sigma} + R^{(2)}_{\mu\kappa} - \frac{1}{2}\eta_{\mu\kappa}R^{(2)}_{\rho\sigma}\eta^{\rho\sigma}}_{\text{quadratic terms as in EM}} + O(h^3)\Big]$$

where
$$R^{(2)}_{\mu\kappa} = -\frac{1}{2}h^{\lambda\nu}\left[h_{\lambda\nu,\kappa\mu} - h_{\mu\nu,\kappa\lambda} - h_{\lambda\kappa,\mu\nu} + h_{\mu\kappa,\nu\lambda}\right]$$
$$+\frac{1}{4}\left(2h^{\nu}{}_{\sigma,\nu} - h^{\nu}{}_{\nu,\sigma}\right)\left(h^{\sigma}{}_{\mu,\kappa} + h^{\sigma}{}_{\kappa,\mu} - h_{\mu\kappa,\sigma}\right)$$
$$-\frac{1}{4}\left(h_{\sigma\kappa,\lambda} + h_{\sigma\lambda,\kappa} - h_{\lambda\kappa,\sigma}\right)\left(h^{\sigma}{}_{\mu,\lambda} + h^{\sigma\lambda}{}_{,\mu} - h^{\lambda}{}_{\mu,\sigma}\right)$$

The presence of third & higher order terms in $t_{\mu\kappa}$ means the self interaction of grav field also contributes to the total energy & momentum.

4) Our construction of $\tau^{\mu\nu}$ clearly suggests that it is the energy-mom tensor produced by the ~~your~~ system as measured of its grav. field. (source of $h_{\mu\nu}$)
There are other definitions of energy-mom tensor of gravitational field.
See later.

5)