9. Gravitational radiation

Recall that in the weak field approx

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 \quad . \tag{9.0.1}$$

Einstein eqn. reads

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{9.0.2}$$

in the gauge $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ Lorentz gauge, sometimes also called the harmonic gauge $\left(g^{\mu\nu} \Gamma^{\lambda}_{\mu\nu} = 0 \text{ in general} \right)$

This eqn. can be solved generally just as in EM.

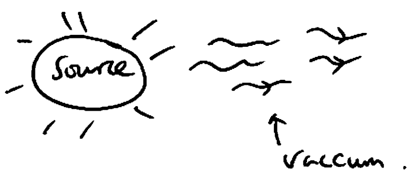
One soln is the retarded potential,

$$\bar{h}_{\mu\nu}(\vec{x}, t) = 4 \int d^3\vec{x}' \, \frac{T_{\mu\nu}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} \tag{9.0.3}$$

← This shows that grav. effects propagate with the speed of light c.

9.1 Plane wave soln.

Consider



propagation of gravitational effects here ~~are~~ are given by soln. to (9.0.3) with $T_{\mu\nu} = 0$.

Thus we need to solve $\Box \bar{h}_{\mu\nu} = 0$ (9.1.1)

This has soln $\bar{h}_{\alpha\beta} = A_{\alpha\beta} e^{ik\cdot x}$ (plane wave)

with $k^2 = k_\mu k^\mu = \eta_{\mu\nu} k^\mu k^\nu = 0$ (9.1.2)

and $\boxed{A^{\alpha\beta} k_\beta = 0}$ (9.1.3)

$A_{\mu\nu} = A_{\nu\mu}$ is called the polarization tensor.

So far we have only the condition (9-1-3) on the Amplitude $A^{\alpha\beta}$. We can use the residual gauge freedom to the Lorentz gauge to further constraint $A^{\alpha\beta}$.

recall Lorentz gauge is achieved by

$$x^\mu \to x^\mu + \xi^\mu$$

with $\Box \xi^\mu = \bar{h}^{(old)\ \mu\nu}{}_{,\nu}$

one can add to ξ^μ any vector η^μ st. $\Box \eta^\mu = 0$ (9.1.4)

This extra gauge transformation leave the Lorentz gauge untouched.

Let us choose a soln to (9.1.4) : $\eta^\mu = B^\mu e^{ik\cdot x}$ for the same k^μ.

This gives $h^{(new)}_{\alpha\beta} = h^{(old)}_{\alpha\beta} - \eta_{\alpha,\beta} - \eta_{\beta,\alpha}$

or $\bar{h}^{(new)}_{\alpha\beta} = \bar{h}^{(old)}_{\alpha\beta} - \eta_{\alpha,\beta} - \eta_{\beta,\alpha} + \eta_{\alpha\beta}\, \eta^\mu{}_{,\mu}$

or $A^{(New)}_{\alpha\beta} = A^{(old)}_{\alpha\beta} - i B_\alpha k_\beta - i B_\beta k_\alpha + i \eta_{\alpha\beta} B^\mu k_\mu$

$\Rightarrow A^{\alpha\ (new)}_{\ \alpha} = A^{\alpha\ (old)}_{\ \alpha} + 2i B^\alpha k_\alpha$

one can always choose B^α st. $\boxed{A^\alpha{}_\alpha = 0}$ (9-1.5)

and $\boxed{A_{\alpha\beta} U^\beta = 0}$ for some fixed timelike 4-vector U^β. (9-1.6)

This is called the transverse-traceless (TT) gauge.

$A^{\alpha\beta}$ satisfying (9.1.3), (9.1.5), (9.1.6)

In this gauge $\bar{h}^{TT}_{\alpha\beta} = h^{TT}_{\alpha\beta}$

We can furthermore go to a Lorentz frame in which the vector u^μ takes the form:

$$u^\mu = \delta^\mu{}_0$$

Then (9.1.6) gives
$$A_{\alpha 0} = 0 \qquad (9.1.7)$$

In this frame, let us further rotate the spatial axis s.t. the wave is travelling in the z-direction : $k = (\omega, 0, 0, \omega)$, then (9.1.3) implies that

$$A_{\alpha z} = 0 \qquad (9.1.8)$$

This means the only nonvanishing components are : $A_{xx} = -A_{yy}$, A_{xy}

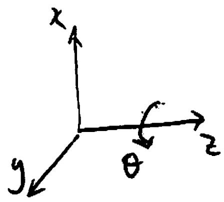
traceless condition (9.1.5)

$$(A^{TT}_{\alpha\beta}) = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad (9.1.9)$$

i.e. only two independent components in $A_{\alpha\beta}$ (or $h_{\alpha\beta}$) !

Helicity

It is interesting to see how $h_{\mu\nu}$ transforms under a rotation.



~~Consider a wave propagating in the~~

Consider a rotation in the z axis. The Lorentz transf. takes the form.

$$\begin{cases} R_1{}^1 = \cos\theta & R_1{}^2 = \sin\theta \\ R_2{}^1 = -\sin\theta & R_2{}^2 = \cos\theta \\ R_3{}^3 = R_0{}^0 = 1 & , \text{ other } R_\mu{}^\nu = 0 \end{cases} \qquad (9.1.10)$$

For a wave propagating in the z-direction $k^\mu = (k, 0, 0, k)$, R leave k inv:

$$R_\mu{}^\nu k_\nu = k_\mu \qquad (9.1.11a)$$

and $h'_{\mu\nu} = R_\mu{}^\rho R_\nu{}^\sigma h_{\rho\sigma}$ for $x^\mu \to x^\mu + \varepsilon^\mu(x)$ (9.1.11b)

For $h_{\mu\nu} = e_{\mu\nu} e^{ikx}$, $e'_{\mu\nu} = e_{\mu\nu} + k_\mu \varepsilon_\nu + k_\nu \varepsilon_\mu$

$k^\alpha e_{\alpha\beta} = \frac{1}{2} k_\beta e^\mu{}_\mu$

If we define $e_\pm \hat{=} e_{11} \mp i e_{12} = -e_{22} \mp i e_{12}$

$$f_\pm \hat{=} e_{31} \mp i e_{32} = -e_{01} \pm i e_{02}$$

$\Rightarrow e_{31} + e_{01} = e_{32} + e_{02} = 0$

$e_{33} + e_{03} = -e_{03} - e_{00}$

$= \frac{1}{2}(e_{11} + e_{22} + e_{33} \mp e_{00})$

then
$$\begin{cases} e'_\pm = e^{\pm 2i\theta} e_\pm \\ f'_\pm = e^{\pm i\theta} f_\pm \\ e'_{33} = e_{33}, \quad e'_{00} = e_{00} \end{cases}$$

In general any plane wave ψ, which is transformed by a rotation of angle θ about the direction of propagation into

$$\psi' = e^{ih\theta}\psi$$

is said to have helicity h.

Therefore we have shown that a grav. wave can be decomposed into

. parts $e_\pm$ with helicity ± 2,

parts $f_\pm$ 〃 〃 ± 1,

parts e_{00}, e_{33} 〃 〃 0.

Moreover, we have shown that by a suitable choice of coord, we can make the helicity $\pm 1, 0$ parts vanishes. Therefore the physically significant components are given by those with $h = \pm 2$.

Due to this fact, we say gravitation are carried by waves of Spin 2!

Instructive to compare this analysis with the Maxwell case:

$$\Box^2 A_\alpha = 0 \quad , \quad \partial_\alpha A^\alpha = 0 \qquad (9.1.12)$$

plane wave soln: $A_\alpha = e_\alpha e^{ik\cdot x}$ $\qquad k_\alpha e^\alpha = 0$, $k_\alpha k^\alpha = 0 \qquad (9.1.13)$

- e^α has 4 components, but $k_\alpha e^\alpha = 0$ reduces it to 3 indep. one.
- gauge transf. $A_\alpha \to A_\alpha + \partial_\alpha \phi$, $\qquad (9.1.14)$

For $\phi = i\varepsilon e^{ikx}$, we get

$$e'_\alpha = e_\alpha - \varepsilon k_\alpha \qquad (9.1.15)$$

This is in analogy with (9.1.16).

we can choose ε to further reduce the components of e_α from 3 to 2.

∴ only two physical degree of freedom in A_α.

For example for a wave in z direction, $k^\mu = (k, 0, 0, k) \Rightarrow e_0 = -e_3$

can choose $e_3 = 0$, then only e_1, e_2 remains.

Finally, for a rotation around the z-axis specified by (9.1.10),

we have $e'_\alpha = R_\alpha{}^\beta e_\beta$

$$\Rightarrow \begin{cases} e'_\pm = e^{\pm i\theta} e_\pm \\ e'_3 = e_3 \end{cases} \qquad e_\pm \triangleq e_1 \mp i e_2 \qquad (9.1.16)$$

therefore the EM wave can be decomposed into parts with helicity ± 1, 0.

The physically significant helicities are ± 1, not 0.

Spin 1.

At this stage, we can study the generation of grav. wave and physical effect of wave.

Let's do the second first.

- generation of grav. wave
- physical effect of grav. wave
  - effect of wave on particles (detector)
  - energy momentum of the wave
  - scattering

Effect of waves on free particles

Consider a free particle initially at rest. It obeys the geodesic eqn.

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0 \qquad (9.1.17)$$

At time $t=0$, $u^\alpha = (1,0,0,0)$ initially at rest.

Moreover $\left.\frac{du^\alpha}{d\tau}\right|_0 = -\Gamma^\alpha_{00} u^0 u^0 = -\frac{1}{2}\eta^{\alpha\beta}(2h_{\beta 0,0} - h_{00,\beta}) + o(h^2)$, (9.1.18)

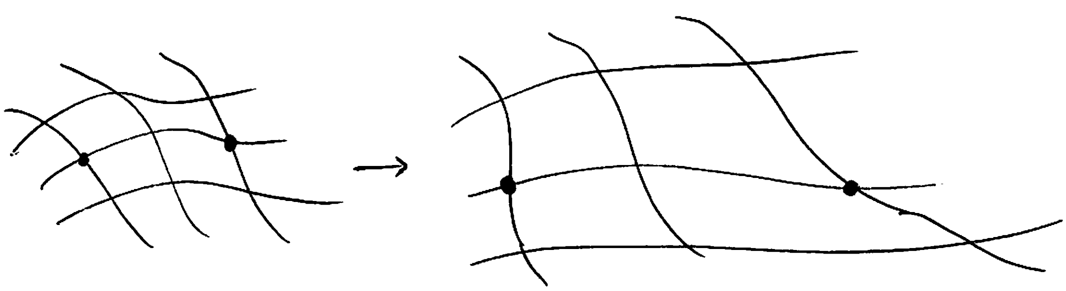
Choose TT gauge, then $h^{TT}_{\beta 0} = 0$

∴ initial acceleration $= 0$

∴ particle remains at rest!

However, being at rest sounds strange as there is a grav. wave acting on the particle!

Correct interpretation : In the TT gauge, a specific adjusment has been made to the coord system such that it stay attached to the individual particle, so it appears that the particle has a constant coord. all the time.



Therefore, to actually tell whether the particle is at rest, we need to measure its motion wrt another free particle.

Let us consider two nearby particles, at $(0,0,0)$ and $(\varepsilon,0,0)$, both initially at rest.

The proper distance between them

$$\Delta l = \int ds = \int |g_{\alpha\beta}\, dx^\alpha dx^\beta|^{1/2} = \int_0^\varepsilon |g_{xx}|^{1/2} dx$$

$$\simeq |g_{xx}(x=0)|^{1/2} \cdot \varepsilon$$

$$\Rightarrow \quad \Delta l \simeq \left[1 + \frac{1}{2} h^{TT}_{xx}(x=0)\right] \varepsilon \qquad (9.1.19)$$

~~Δl~~ change with time since h^{TT}_{xx} is generally not zero.

1° Notice that : coord. position is time indep.

proper distance is time dep.

unphysical as it is coord dep.

physical as it is diffeo inv.

2° (9.1.19) gives a mean to measure the effect of grav. wave.

- In particular, the measurable effect Δl is bigger if initial separation is bigger.

  Hence why grav. wave detectors are huge in size!

- the effect is small as h^{TT}_{ij} tends to be tiny. (eg. 10^{-21})

  The magnitude of $h_{\mu\nu}$ depends on its source (generation ← (9.0-3) and propagation through space. see later.

Tidal forces

we can also measure the effect of grav. wave (or generally of grav.) by measuring the tidal forces. This is determined by the eqn. of geodesic deviation:

$$\nabla_V \nabla_V \xi^\alpha = R^\alpha{}_{\mu\nu\beta} V^\mu V^\nu \xi^\beta \qquad \text{recall} \left(\nabla_V \xi^\alpha = V^\beta \xi^\alpha{}_{;\beta} \right)$$

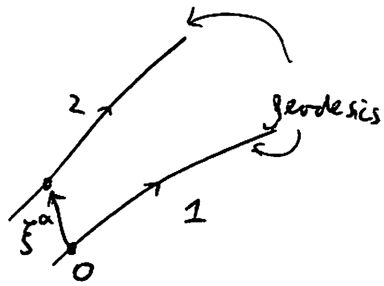
(9.1.20)

- we can apply this eqn. in any frame, but we better not in the TT ~~Gauge~~ -Coord. System since we have already seen the coordinates of the particles remain constant. Hence the connecting vector ξ^α remains constant.

- Let us use (9.1.20) in the local inertial frame at the point O of the geodesic of first particle. (ie normal Coord. System)

Consequences:

i ~~AS~~ locally $g_{\mu\nu} = \eta_{\mu\nu}$,



$\therefore$ ξ^α (or its length) does correspond to measurable proper distance if the geodesics are close enough.

ii. Take U^μ to be the tangent vector at O, then "simplifications":

$$\nabla_u \xi^\alpha = \frac{d\xi^\alpha}{d\tau}$$

$$\nabla_u \nabla_u \xi^\alpha = \nabla_u \frac{d\xi^\alpha}{d\tau} = \frac{d^2\xi^\alpha}{d\tau^2} \qquad \because \Gamma^\alpha_{\beta\gamma} \text{ vanishes also locally!}$$

$\therefore$ (9.1.20) reads

$$\frac{d^2\xi^\alpha}{d\tau^2} = R^\alpha{}_{\mu\nu\beta} U^\mu U^\nu \xi^\beta \qquad (9.1.21)$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity of the 1st particle.

Now initially at rest $\Rightarrow$ $u^\mu = (1,0,0,0)$

Also initial separation $\xi^\mu = (0, \varepsilon, 0, 0)$

$$\therefore \left.\frac{d^2 \xi^\alpha}{d\tau^2}\right|_0 \cong \left.\frac{\partial^2 \xi^\alpha}{\partial t^2}\right|_0 = \varepsilon R^\alpha{}_{00x} = -\varepsilon R^\alpha{}_{0x0} \qquad (9.1.22)$$

$\therefore$ Riemann tensor (at least some of its components) is locally measurable, by using the normal coord. system, by watching the change of proper distance between nearby geodesics.

- The relation (9.1.21) was derived in the ~~be~~ normal coord system. The components of $R^\mu{}_{\nu\alpha\beta}$ was computed in this coord system. However since we are interested in leading order terms for small $h_{\mu\nu}$. We can also compute $R^\mu{}_{\nu\alpha\beta}$ in the TT-coord. system. The difference is of higher order in h.

In the TT-coord., $R^x{}_{0x0} = R_{x0x0} = -\frac{1}{2} h^{TT}_{xx,00}$

$$R^y{}_{0x0} = R_{y0x0} = -\frac{1}{2} h^{TT}_{xy,00}$$

$$R^y{}_{0y0} = R_{y0y0} = -\frac{1}{2} h^{TT}_{yy,00} = -R^x{}_{0x0}$$

and all other independent components vanishes.

$$\left(\text{recall } R_{\alpha\beta\mu\nu} \simeq \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}\right) \right)$$

As a result,

$$\begin{cases} \dfrac{\partial^2 \xi^x}{\partial t^2} = \dfrac{1}{2}\varepsilon \dfrac{\partial^2}{\partial t^2} h_{xx}^{TT} \\[2ex] \dfrac{\partial^2 \xi^y}{\partial t^2} = \dfrac{1}{2}\varepsilon \dfrac{\partial^2}{\partial t^2} h_{xy}^{TT} \end{cases} \tag{9.1.23}$$

These are consistent with (9.1.19)

Similarly, for two particles initially separated by $y=\varepsilon$, we have

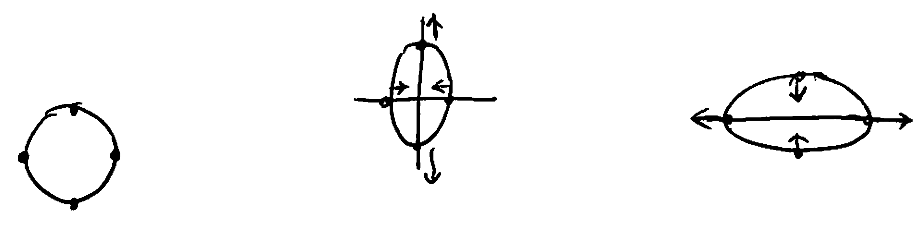
$$\begin{cases} \dfrac{\partial^2 \xi^y}{\partial t^2} = \dfrac{1}{2}\varepsilon \dfrac{\partial^2 h_{yy}^{TT}}{\partial t^2} = -\dfrac{1}{2}\varepsilon \dfrac{\partial^2 h_{xx}^{TT}}{\partial t^2} \\[2ex] \dfrac{\partial^2 \xi^x}{\partial t^2} = \dfrac{1}{2}\varepsilon \dfrac{\partial^2}{\partial t} h_{xy}^{TT} \end{cases} \tag{9.1.24}$$

- (9.1.23) + (9.1.24) gives two indep. modes of motion, called polarizations

mode 1: $h_{xx}^{TT} \neq 0, \quad h_{yx}^{TT} = 0$

$\Delta x \neq 0$: $\begin{cases} \dfrac{\partial^2 \xi^x}{\partial t^2} = \dfrac{1}{2}\Delta x \dfrac{\partial^2 h_{xx}^{TT}}{\partial t^2} \\[2ex] \dfrac{\partial^2 \xi^y}{\partial t^2} = 0 \end{cases}$

$\Delta y \neq 0$: $\begin{cases} \dfrac{\partial^2 \xi^x}{\partial t^2} = 0 \\[2ex] \dfrac{\partial^2 \xi^y}{\partial t^2} = -\dfrac{1}{2}\Delta y \dfrac{\partial^2 h_{xx}^{TT}}{\partial t^2} \end{cases}$

$\Rightarrow$ 

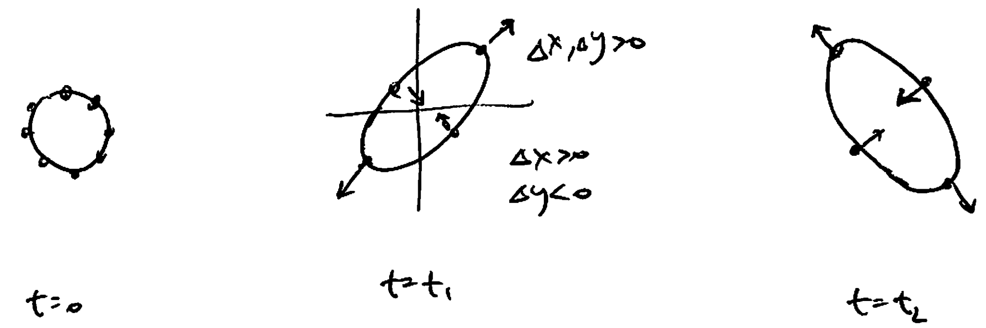
$t=0$ $\qquad$ $t=t_1$ $\qquad$ $t=t_2$

mode 2: $h_{xx}^{TT} = 0$, $h_{xy}^{TT} \neq 0$

$$\Delta x \neq 0 : \begin{cases} \dfrac{\partial^2 \xi^x}{\partial t^2} = 0 \\ \dfrac{\partial^2 \xi^y}{\partial t^2} = \dfrac{1}{2}\Delta x \dfrac{\partial^2 h_{xy}^{TT}}{\partial t^2} \end{cases}$$

$$\Delta y \neq 0 : \begin{cases} \dfrac{\partial^2 \xi^x}{\partial t^2} = \dfrac{1}{2}\Delta y \dfrac{\partial^2 h_{xy}^{TT}}{\partial t^2} \\ \dfrac{\partial^2 \xi^y}{\partial t^2} = 0 \end{cases}$$

$\Rightarrow$



NB. The polarizations are different from EM polarizations (45° vs. 90° relative to each other)

This is because grav. wave is carried by a 2nd rank symm. tensor instead of a first rank tensor.

An exact soln: plane wave

It turns out one can obtain an Exact soln of Einstein eqn., it is given by a plane wave. Consider a wave moving in the z-direction. Rare!

The soln must depend only on $u \triangleq t - z$

Introduced also $v = t + z$

then flat metric $ds^2 = -du\,dv + dx^2 + dy^2$ (9.1.23)

For a plane wave, we consider the ansatz

$$ds^2 = -du\,dv + f^2(u)\,dx^2 + g^2(u)\,dy^2 \quad (9.1.24)$$

Christoffel symbol :

$$\begin{cases} \Gamma^x_{xu} = \dot{f}/f \,, & \Gamma^y_{yu} = \dot{g}/g \\ \Gamma^v_{xx} = 2\dot{f}/f \,, & \Gamma^v_{yy} = 2\dot{g}/g \\ R^x{}_{uxu} = -\ddot{f}/f \,, & R^y{}_{uyu} = -\ddot{g}/g \\ \text{others} = 0 & \end{cases} \quad (9.1.25)$$

Vaccum field eqn: $R_{\mu\nu} = 0$

$$\Rightarrow \quad \ddot{f}/f + \ddot{g}/g = 0 \qquad \Rightarrow \text{ solve } g \text{ in terms of } f. \quad (9.1.26)$$

Thus we obtain a general soln. given in term of an arb. fun. $f(u)$.

eg. $f = 1 - \varepsilon(u)$, $g \simeq 1 + \varepsilon(u)$ for small $|\varepsilon| \ll 1$

This corresponds to the linear wave with $h_{xy} = 0$.

9.2 Generation of gravitational wave

The wave eqn.

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

has ~~general~~ retarded soln.

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4\int \frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}\,d^3y \qquad (9.2.1)$$

← Note: Not in TT gauge

~~For~~ Far field, ~~approx~~ $|x^i| \equiv r \gg |y^i| \equiv y$

Can approx (9.2.1) by

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4}{r}\int T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})\,d^3y \qquad (9.2.2)$$

Conservation law $T^{\mu\nu}{}_{,\nu}=0$

$$\Rightarrow \quad T^{0\mu}{}_{,0} + T^{i\mu}{}_{,i} = 0$$

$$\int d^3y \Rightarrow \frac{d}{dt}\int T^{0\mu}\,d^3y + \underbrace{\int \partial_i T^{i\mu}\,d^3y}_{0} = 0$$

$$\Rightarrow \int T^{0\mu}\,d^3y = \text{time indep.}$$

$\therefore \bar{h}^{0\mu}$ is time indep. $\Rightarrow$ does not contribute to a wave field.

Next, $\displaystyle \frac{d^2}{dt^2}\int T^{00}x^l x^m\,d^3x = 2\int T^{lm}\,d^3x$ (tensor virial theorem)

Pf. $\displaystyle \frac{d}{dt}\int T^{00}x^l x^m\,d^3x = -\int \partial_i T^{0i}x^l x^m\,d^3x$

$$= +\int T^{0i}(\delta^l_i x^m + \delta^m_i x^l)\,d^3x$$

$$\frac{d^2}{dt^2}\int T^{00} x^\ell x^m d^3x = \int \partial_0 T^{0i}(\delta^\ell_i x^m + \delta^m_i x^\ell)\, d^3x$$

$$= \int T^{ji}(\delta^\ell_i \delta^m_j + \delta^m_i \delta^\ell_j)\, d^3x$$

$$= 2\int T^{\ell m} d^3x$$

//

Define $I^{\ell m} = \int T^{00} x^\ell x^m d^3x$ quadrupole moment tensor of mass distribution. (9.2.3)

then $\bar{h}_{jk} = \frac{4}{r}\underbrace{\int T_{jk}(t-|\vec{x}-\vec{y}|,\vec{y})\, d^3y}_{= \frac{1}{2} I_{jk,00}}$

$$= \frac{2}{r} I_{jk}(t-r)_{,00}$$

check gauge conditions: $\partial^\mu \bar{h}_{\mu\nu} = 0$?

$\cancel{\partial^0 \bar{h}_{0\nu} + \partial^i \bar{h}_{i\nu}}$

$\cancel{\text{for } \nu = j: \ \frac{2}{r}\int \partial^i I_{ij}}$

(9.2.2) : $\partial^\mu \bar{h}_{\mu\nu} = \frac{2}{r}\int \partial^\mu T_{\mu\nu}\, d^3y = 0$ ✓

$\bar{h}^\alpha{}_\alpha = 0$?

$$\bar{h}^0{}_0 + \bar{h}^j{}_j = \bar{h}^0{}_0 + \frac{2}{r} I^j{}_{j,00} \neq 0$$

time indep in general. $\neq 0$

To go to a TT gauge, we perform a gauge transf.

which amount to adding a trace to the $\bar{h}_{\mu\nu}$:

$$\bar{h}^{(old)}_{\mu\nu} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

$$\bar{h}^{(new)} = \begin{pmatrix} a+\delta & c \\ c & b+\delta \end{pmatrix}$$

To achieve traceless condition, we take $2\delta + a + b = 0$

$$\Rightarrow \delta = -\left(\frac{a+b}{2}\right)$$

then $\bar{h}^{(new)} = \begin{pmatrix} \frac{a-b}{2} & c \\ c & -\frac{a-b}{2} \end{pmatrix}$

In the TT gauge,

$$\bar{h}^{(TT)}_{xx} = \frac{2}{r}\left(I_{xx,00} - I_{yy,00}\right) = -\bar{h}^{(TT)}_{yy}$$

$$\bar{h}^{(TT)}_{xy} = \frac{2}{r}\, I_{xy,00}$$

we can also use $\mathcal{I}_{ij} \stackrel{\circ}{=} I_{ij} - \frac{1}{3}\delta_{ij} I^{k}{}_{k}$

then
$$\begin{cases} \bar{h}^{(TT)}_{xx} = -\bar{h}^{(TT)}_{yy} = \frac{2}{r}\left(\mathcal{I}_{xx,00} - \mathcal{I}_{yy,00}\right) \\ \bar{h}^{(TT)}_{xy} = \frac{2}{r}\, \mathcal{I}_{xy,00} \end{cases} \qquad (9.2.4)$$

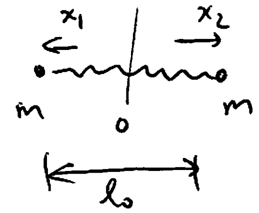
- For a source $T_{\mu\nu} = S_{\mu\nu}(x^i)\, e^{-i\Omega t}$, we have

TT gauge
$$\begin{cases} \bar{h}_{zi} = 0 \\ \bar{h}_{xy} = -\cancel{2}\,\Omega^2 \frac{e^{i\Omega t}}{r}\, \mathcal{I}_{xy} \\ \bar{h}_{xx} = -\bar{h}_{yy} = -\Omega^2\left(\mathcal{I}_{xx} - \mathcal{I}_{yy}\right)\frac{e^{i\Omega t}}{r} \end{cases} \qquad (9.2.5)$$

$\mathcal{I}_{ij}$ has T_{ij} replaced by S_{ij}.

Examples:

simple harmonic oscillator



equilibrium

$$\begin{cases} x_1 = -\frac{l_0}{2} - A\cos\omega t \\ x_2 = \frac{l_0}{2} + A\cos\omega t \end{cases}$$

$$I_{xx} = m(x_1^2 + x_2^2) = \underbrace{\underline{\text{const.}} + \underline{mA^2\cos 2\omega t} + \underline{2ml_0A\cos\omega t}}_{\text{3 components.}} \qquad (9\text{-}2\text{-}6)$$

1st component: no wave

3rd components:

$$\bar{I}_{xx} = I_{xx} - \frac{1}{3}I^i_{\ j} = \frac{2}{3}I_{xx} = \frac{4}{3}ml_0Ae^{-i\omega t}$$

$$\bar{I}_{yy} = \bar{I}_{zz} = -\frac{1}{3}I_{xx} = -\frac{2}{3}ml_0Ae^{-i\omega t}$$

for radiation in the z direction,

$$\Rightarrow \begin{cases} \bar{h}^{TT}_{xx} = -\bar{h}^{TT}_{yy} = -2m\omega^2 l_0 A\,\dfrac{e^{i\omega(r-t)}}{r} \\ \bar{h}^{TT}_{xy} = 0 \end{cases} \qquad (9\text{-}2\text{-}7)$$

2nd component:

$$\begin{cases} \bar{h}^{TT}_{xx} = \bar{h}^{TT}_{yy} = -8m\omega^2A^2\,\dfrac{e^{2i\omega(r-t)}}{r} \\ \bar{h}^{TT}_{xy} = 0 \end{cases} \qquad (9\text{-}2\text{-}8)$$

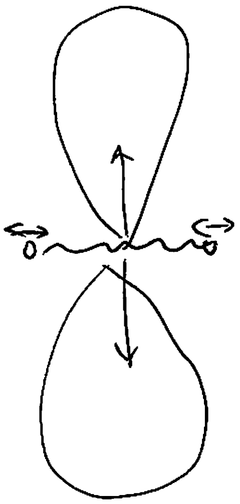
Total radiation field: $\bar{h}_{xx} = \frac{-1}{r}\left[2m\omega^2 l_0 A\cos\omega(r-t) + 4m\omega^2A^2\cos 2(\omega(r-t))\right]$

etc.

However for radiation in the x direction,

$$\begin{cases} \bar{h}_{yy} = -\bar{h}_{zz} = 0 & \propto (I_{yy} - I_{zz}) \\ \bar{h}_{yz} = 0 \end{cases}$$

No radiation in the x direction

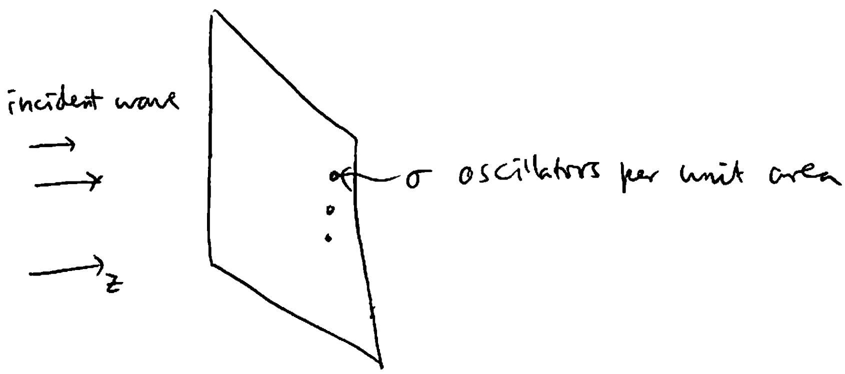


← zero radiation !

9.3 Energy carried by grav. wave

Q. What is the energy flux F for grav. wave ?

We can derive for for a plane wave by considering the set up.



(9.3.1)

Incident wave, in TT gauge,

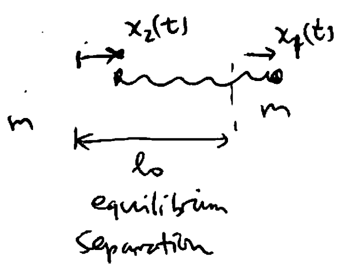
$$\begin{cases} \bar{h}^{TT}_{xx} = A\cos\Omega(z-t) = -\bar{h}^{TT}_{yy} \\ \bar{h}^{TT}_{xy} = 0 \end{cases} \qquad (9.3.2)$$

This set off steady oscillation $\xi = R\cos(\Omega t + \phi)$

R: amplitude, ϕ: phase factor.

details unimportant for us

Basically, ~~the~~ each oscillator in (9.3.1) is a set of two identical masses attached by a spring of spring constant k and a damping constant ν.



EoM :

$$\begin{cases} m\ddot{x}_1 = -k(x_1 - x_2 + l_0) - \nu\frac{d}{dt}(x_1 - x_2) \\ m\ddot{x}_2 = -k(x_2 - x_1 - l_0) - \nu\frac{d}{dt}(x_2 - x_1) \end{cases} \qquad (9.3.2)$$

If we define $\xi = x_2 - x_1 - l_0$, $\omega_0^2 = 2\frac{k}{m}$, $\gamma = \nu/m$

then $\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = 0$ (9-3-3)

In the presence of a grav. wave, $x_2 - x_1$ should be replaced by

$$\Delta X(t) = \int_{x_1(t)}^{x_2(t)} \sqrt{1 + h_{xx}^{TT}(t)}\, dx = (x_2 - x_1)\left(1 + \frac{1}{2} h_{xx}^{TT}(t)\right) + O(|h|^2)$$

(9.3.2) should be replaced by :

$$\begin{cases} m\ddot{x}_1 = -k(l_0 - \Delta X) - \nu\frac{d}{dt}(l_0 - \Delta X) \\ m\ddot{x}_2 = -k(\Delta X - l_0) - \nu\frac{d}{dt}(\Delta X - l_0) \end{cases}$$

If we define $\xi = \Delta X - l_0$, then we obtain

$$\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = \frac{1}{2} l_0 h_{xx,00}^{TT} \qquad (9.3.4)$$

correct to first order in h_{xx}^{TT}.

For a wave of the form $h_{xx}^{TT} = A\cos\Omega t$,

(9.3.4) is just a damped oscillator with an external force,

so we obtain

$$\xi = R\cos(\Omega t + \phi) \qquad R = \frac{l_0\Omega^2 A}{2}\frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\Omega^2\gamma^2}}$$

$\tan\phi = 2\gamma\Omega/(\omega_0^2 - \Omega^2)$

Assuming the motion is steady, meaning that the energy dissipated by friction is exactly compensated by the energy input by the tidal force of the grav. wave, then

$$\frac{dE}{dt} = \nu \left(\frac{d\xi}{dt}\right)^2 = m\gamma \left(\frac{d\xi}{dt}\right)^2$$

$$\Rightarrow \left\langle \frac{dE}{dt} \right\rangle = \frac{1}{2} m \gamma \Omega^2 R^2$$

With σ oscillators per unit area, the energy flux F, must decrease on passing the plane of oscillators:

$$\delta F = -\frac{1}{2}\sigma m \gamma \Omega^2 R^2 \qquad (9.3.5)$$

As energy is being drawn from the wave, we can expect that the amplitude of the wave will decrease.

Now each oscillator has a quadrupole tensor given by (9.2.6):

$$I_{xx} = \text{const.} + mA^2 \cos 2\omega t + 2ml_0 A \cos \omega t$$

with ωt replaced by $\Omega t + \phi$

A replaced by $R/2$

Since R is tiny $\ll l_0$, we can ignore the 2ω terms and so for each oscillator,

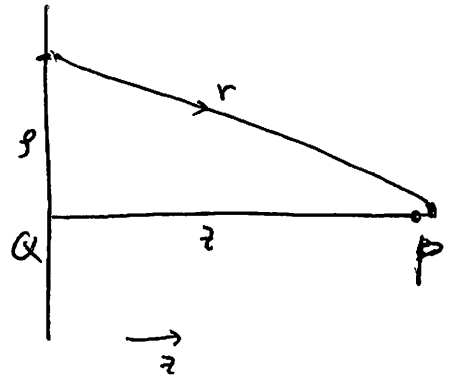
$$I_{xx} = m l_0 R \cos(\Omega t + \phi) \qquad (9.3.6)$$

Each oscillator produces a wave amplitude, see (9.2.7),

$$\delta \bar{h}_{xx} = -\frac{\Omega^2 m l_0 R}{r} \cos\left[\Omega(r-t) - \phi\right]$$

(factor of 2 error in book!)

To get the total contribution from the whole plane of oscillators, consider a osillator at distance ρ from the point Q,



observation point

$$\delta \bar{h}^{\text{total}}_{xx} = -m\Omega^2 l_0 R \int_0^\infty \frac{\sigma \cos \Omega[(r-t) - \phi]}{r} 2\pi \rho \, d\rho$$

Now $\rho^2 + z^2 = r^2 \Rightarrow \rho \, d\rho = r \, dr$

$$\therefore \delta \bar{h}^{\text{total}}_{xx} = -m\Omega^2 l_0 R \int_z^\infty \sigma \cos\left[\Omega(r-t) - \phi\right] dr \qquad (9.3.7)$$

If σ = constant, this gives ∞ and does not make sense.

To get sensible result, ~~replace~~ regulate σ by $\sigma e^{-\varepsilon r}$ ($\because$ expect distant oscillator to not to contribute)

then $$\delta \bar{h}^{\text{total}}_{xx} = \underbrace{2\pi\sigma\Omega m l_0 R}_{\equiv \delta} \sin\left[\Omega(r-t) - \phi\right] \qquad (9.3.8)$$

Add this to the incident wave, we get

$$\bar{h}_{xx}^{net} = \left[A - 2\pi\sigma m \Omega l_0 R \sin\phi\right] \cos\left[\Omega(z-t) - \psi\right] + o(R^2)$$

$$\therefore \delta A = -2\pi\sigma m \Omega l_0 R \sin\phi \qquad (9.3.9)$$

$$A\cos\Omega(z-t) + \delta\sin[\Omega(z-t) - \phi]$$

$$= \cos\Omega(z-t) \times [A - \delta\sin\phi] + \sin\Omega(z-t) \times \delta\cos\phi$$

$$= \cos[\Omega(z-t) - \psi] \times D$$

$$D = \sqrt{(A - \delta\sin\phi)^2 + (\delta\cos\phi)^2}$$

$$\tan\psi = \frac{\delta\cos\phi}{A - \delta\sin\phi}$$

In particular, we obtain

$$\frac{\delta F}{\delta A} = +\frac{1}{2} m\sigma\gamma\Omega^2 R^2 \frac{1}{2\pi\sigma m\Omega l_0 R\sin\phi}$$

$$= \frac{\gamma}{4\pi}\Omega \underbrace{\frac{R}{l_0\sin\phi}}_{\frac{\Omega^2 A}{2}\frac{1}{\sqrt{(\omega_0^2-\Omega^2)^2 + 4\Omega^2\gamma^2}}\cdot\frac{\sqrt{}}{2\gamma\Omega}}$$

$$\frac{\delta F}{\delta A} = \frac{1}{16\pi}\Omega^2 A \qquad (9.3.10)$$

It is remarkable that the result is independent of any details of the oscillator (m, γ, ω_0^2) etc.

It simply says a change δA in the amplitude of the wave of freq Ω changes its flux F by the amount (9.3.10)!

Integrating (9.3.10) $\Rightarrow$ $F = \frac{1}{32\pi}\Omega^2 A^2$ (9.3.11)

since $\langle(\bar{h}^{TT}_{xx})^2\rangle = \frac{1}{2}A^2$

We have

$$F = \frac{1}{16\pi^2}\,\Omega^2 \left\langle \bar{h}^{TT}_{\mu\nu}\,\bar{h}^{TT\,\mu\nu} \right\rangle \qquad (9.3\text{-}12)$$

written in an inv. form under Background Lorentz transf.

(But not under gauge transf.!)

(9.3.12) is indeed true for any wave front, not just plane wave.

<u>Application</u> : <u>Energy lost by a grav. radiating system</u>

Rewriting (9.3.12),

$$F = \frac{\Omega^6}{32\pi r^2}\left\langle \left(2\left(\mathcal{I}_{xx} - \mathcal{I}_{yy}\right)^2 + 8\,\mathcal{I}^2_{xy} \right) \right\rangle \qquad \text{using (9.2.5)}$$

Using $\mathcal{I}^i{}_i = 0 = \mathcal{I}_{xx} + \mathcal{I}_{yy} + \mathcal{I}_{zz}$

$$\Rightarrow \quad F = \frac{\Omega^6}{16\pi r^2}\left\langle 2\,\mathcal{I}_{ij}\mathcal{I}^{ij} - 4\,\mathcal{I}_{zj}\mathcal{I}_z{}^j + \mathcal{I}^2_{zz} \right\rangle$$

Covariantizing $n^z = 1 \to n^i = x^i/r$

get $F = \dfrac{\Omega^6}{16\pi r^2}\left\langle 2\,\mathcal{I}_{ij}\mathcal{I}^{ij} - 4 n^j n^k\,\mathcal{I}_{ji}\mathcal{I}_k{}^i + n^i n^j n^k n^l\,\mathcal{I}_{ij}\mathcal{I}_{kl} \right\rangle$

Total Luminosity $\quad L = \displaystyle\int F\, r^2 \sin^2\theta\, d\theta\, d\phi$

$$\int n^j n^k \sin\theta \, d\theta \, d\phi = \frac{4\pi}{3}\delta^{jk}$$

$$\int n^i n^j n^k n^l \sin\theta \, d\theta \, d\phi = \frac{4\pi}{15}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$$

$$\Rightarrow L = \frac{\omega^6}{5}\langle \Xi_{ij} \Xi^{ij} \rangle$$

This is for a single freq.

For general time dep., we have,

$$L = \frac{2}{5}\langle \dddot{\Xi}_{ij} \dddot{\Xi}_{ij} \rangle \qquad (9.3\text{-}13)$$

Application of (9.3.13), energy loss by binary star system.