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Solution
Final Examination
Introduction to Relativity I (PHYS431000)

1. (Fundamental concepts)

(a) State the Einstein equivalence principle.
Soln: At every spacetime point in an arbitrary gravitational field, it is possible to choose a locally inertial coordinate system given by a locally free falling observer such that within a sufficiently small region, the laws of nature take the same form as in an unaccelerated Cartesian coordinate system in the absence of gravity.

[8pts]

(b) Explain how Einstein equivalence principle leads to the presence of gravitational redshift.
Soln: See my note p.5.3. Key equations are (5.1.5) and (5.1.6). *[12pts]*

2. (Deriving Einstein Equation)

(a) Explain why spacetime must be curved in the presence of gravity.
Soln: See my note p.5.4. *[8pts]*

(b) Assume that the law of gravitation take the tensorial form

$$K_{\mu\nu} = kT_{\mu\nu}$$

where k is a constant, $Y_{\mu\nu}$ is the energy momentum tensor and $K_{\mu\nu}$ is a tensor. State the conditions which $K_{\mu\nu}$ should satisfy.
Soln:
$K_{\mu\nu}$ must be symmetric. $T_{\mu\nu}$ is covariantly conserved, implying that ${K^{\mu\nu}}_{;\nu} = 0$. *[4pts]*

(c) State the Bianchi identity for the Riemann curvature tensor ${R^\lambda}_{\mu\alpha\beta}$. Use it to derive the unique combination, up to an overall constant, of $K_{\mu\nu}$ in terms of $R_{\mu\nu}$ and $g_{\mu\nu}R$.
Soln:
Bianchi identity:

$$R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\beta\gamma;\alpha} + R_{\mu\nu\gamma\alpha;\beta} = 0.$$

Multiply by $g^{\mu\alpha}$, we get

$$R_{\nu\beta;\gamma} + {R^\alpha}_{\nu\beta\gamma;\alpha} - R_{\nu\gamma;\beta} = 0.$$

Contract with $g^{\nu\beta}$,

$$R_{;\gamma} - {R^\alpha}_{\gamma;\alpha} - {R^\beta}_{\gamma;\beta} = 0.$$

i.e.

$$(2{R^\beta}_\gamma - {\delta^\beta}_\gamma R)_{;\beta} = 0.$$

Hence up to a multiplicative constant,

$$K_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

[8pts]

3. (Newtonian Limit)

(a) What is the generalization of the Newton's first law of motion in general relativity?
Soln:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = f^\mu/m,$$

where f^μ is the four-force. *[8pts]*

(b) *Consider the case where there is no other external force other than the presence of a gravitational force. State the conditions for Newtonian limit.*

Soln: Newtonian limit is when $|\phi| \ll 1$ *and* $v/c \ll 1$.

[4pts]

(c) *Show that in the Newtonian limit, the equation of the motion in (a) reduces to the ordinary Newton's law of gravity.*

Soln: In the absence of gravity, spacetime is flat and we use rectangular coordinates

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z.$$

In the presence of a wealk gravitational field,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

For small velocities,

$$(ds/d\tau)^2 = (\eta_{\mu\nu} + h_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \approx (\frac{dt}{d\tau})^2$$

The geodesic equation becomes

$$\frac{d^2x^\mu}{ds^2} \approx -\Gamma^\mu_{00}(\frac{dx^0}{ds})^2 \approx -\Gamma^\mu_{00}.$$

As for Γ^μ_{00}, *we find*

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\mu}(\frac{\partial h_{00}}{\partial x^\mu} - 2\frac{\partial h_{\mu 0}}{\partial x^0})$$

Ignoring the time derivative term for slow speeds, we get

$$\Gamma^\mu_{00} \approx \frac{1}{2}\eta^{\mu\mu}\frac{\partial h_{00}}{\partial x^\mu}$$

and

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$$

This is the form of the Newtonian gravity

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i}$$

if

$$h_{00} = 2\Phi.$$

[8pts]

4. (Gravitation Wave)

(a) *Consider a weak gravitational field*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1.$$

Derive the transformation of $h_{\mu\nu}$ *under an infinitesimal coordinate transformation*

$$x^\mu \to x^m + \xi^\mu.$$

(b) *Einstein equation in a weak gravitational field*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

can be written as

$$\Box \hat{h}_{\mu\nu} = -16\pi G T_{\mu\nu} + \theta_{\mu\nu},$$

where

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha}$$

and

$$\theta_{\mu\nu} = \partial_\nu \partial^\alpha \hat{h}_{\mu\alpha} + \partial_\mu \partial^\alpha \hat{h}_{\nu\alpha} - \eta_{\mu\nu}\partial^\alpha \partial^\beta \hat{h}_{\alpha\beta}$$

Show that one can always go to a certain specific coordinate system such that the $\theta^{\mu\nu}$ term vanishes and the weak field Einstein equation reads

$$\Box \hat{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

(c) Consider the propagation of gravitational wave in empty space. The wave can be represented as

$$\hat{h}_{\mu\nu} = Re(A_{\mu\nu}e^{ik_\alpha x^\alpha}).$$

Show that k_α must be a null vector and that

$$A_{\mu\nu}k^\nu = 0.$$

5. (Geodesic Equation)

(a) Show that the geodesic equation can be written as

$$\frac{d}{ds}(g_{\mu\nu}\frac{dx^\nu}{ds}) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

(b) Consider the metric outside a blackhole

$$ds^2 = -(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Show that

$$r^2\frac{d\varphi}{ds} = h = a\ constant.$$

$$(1-\frac{2m}{r})\frac{dt}{ds} = l = a\ constant.$$

(c) Show that for a massless photon, it follows the trajectory

$$\frac{d^2u}{d\varphi^2} + u = 3u^2,$$

where

$$u = m/r.$$