

Solution
Mid Examination
Introduction to Relativity I (PHYS431000)

1. (Baic concepts)

- (a) State the fundamental postulates of special relativity.

Soln: The law of nature and the results of all experiments performed in a given frame of reference are independent of the uniform translational motion of the system as a whole.

The velocity of light is independent of the uniform translational motion of the source. i.e. Unaccelerated observers will find the speed of light to be the same constant. [10pts]

- (b) Mr Ma says he discovers a new physical law about time travel and it is written in the form

$$T^{\mu\nu} + H_{\lambda}^{\mu\nu} S^{\lambda} = E_{\mu}{}^{\nu},$$

where T, H, S, E are tensors. Explain why this cannot be correct.

Soln: Physical laws should be form invariant under Lorentz transformation. This can be most conveniently implemented by writing the physical laws in tensorial form. The given equation is not a valid tensor equation since the LHS and the RHS are tensors of different type and so transformed differently under Lorentz transformation. In other word, the given equation is not form invariant under Lorentz transformation. [5pts]

- (c) Mr Ba Ba says he discovers a new 4-force
- $f^{\mu}(x^{\nu})$
- which depends only on the position vector
- x^{ν}
- of a particle, but not its 4-velocity
- u^{ν}
- . Show that this is not consistent with special relativity.
- Hint: in special relativity, a force f^{μ} induces a 4-acceleration*

$$a^{\mu} = \frac{du^{\mu}}{d\tau} = \frac{1}{m} f^{\mu}.$$

Soln: For a non-zero 4-force f independent of u , there must exist some 4-velocity such that $f_{\mu}u^{\mu}$ is nonzero. [5pts]

However, by definition of 4-velocity, we have $u^{\mu}u_{\mu} = -1$. Differentiating with respect to τ , we get

$$u_{\mu}a^{\mu} = 0.$$

This implies in general the 4-force is always orthogonal to the 4-velocity

$$u_{\mu}f^{\mu} = 0.$$

This is a contradiction. [5pts]

2. (Lorentz transformations)

- (a) Show that two successive Lorentz transformations in the same direction (say
- x
-) commute and that they are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}$$

Soln: Use

$$t' = \frac{t - v_1 x}{\sqrt{1 - v_1^2}}, \quad x' = \frac{x - v_1 t}{\sqrt{1 - v_1^2}}$$

$$t'' = \frac{t' - v_2 x'}{\sqrt{1 - v_2^2}}, \quad x'' = \frac{x' - v_2 t'}{\sqrt{1 - v_2^2}}$$

[4pts]

One find

$$t'' = \frac{t - vx}{\sqrt{1 - v^2}}, \quad x'' = \frac{x - vt}{\sqrt{1 - v^2}}$$

with the given v .

[5pts]

- (b) Show that two successive Loentz transformation at right angles (v_1 in the x direction, v_2 in the y direction) do not commute. i.e. show that the boosts performed in the reverse order gives a different transformation.

Soln: perform v_x boost and then v_y boost:

$$\begin{aligned} t' &= \gamma_x(t - v_x x), & x' &= \gamma_x(x - v_x t), & y' &= y \\ t'' &= \gamma_y(t' - v_y y'), & y'' &= \gamma_y(y' - v_y t'), & x'' &= x', \end{aligned}$$

[4pts]

we get

$$t'' = \gamma_x \gamma_y t - \gamma_x \gamma_y v_x x - \gamma_y v_y y, \quad x'' = -\gamma_x v_x t + \gamma_x x, \quad y'' = -\gamma_x \gamma_y v_y t + \gamma_x \gamma_y v_x v_y x + \gamma_y y.$$

[4pts]

On the other hand, perform v_y boost first and then v_x boost:

$$\begin{aligned} t' &= \gamma_y(t - v_y y), & y' &= \gamma_y(y - v_y t), & x' &= x, \\ t'' &= \gamma_x(t' - v_x x'), & x'' &= \gamma_x(x' - v_x t'), & y'' &= y' \end{aligned}$$

[4pts]

we get

$$t'' = \gamma_x \gamma_y t - \gamma_x v_x x - \gamma_x \gamma_y v_y y, \quad x'' = -\gamma_x \gamma_y v_x t + \gamma_x x + \gamma_x \gamma_y v_x v_y y, \quad y'' = -\gamma_y v_y t + \gamma_y y.$$

[4pts]

The two are different.

3. (Scattering of photon and electron)

- (a) Photon is the quanta of light. It has zero mass. According to quantum mechanics, a photon of wavelength λ carries a momentum of the magnitude

$$p = \frac{h}{\lambda},$$

where h is the Planck constant. Consider a photon of wavelength λ hits a stationary electron (mass m_e) and comes off with wavelength λ' at an angle θ . This scattering process is called the Compton scattering. Derive the expression

$$\lambda' - \lambda = \frac{h}{m_e}(1 - \cos \theta)$$

Soln: 4-momentum conservation

$$p_e + p_\gamma = p'_e + p'_\gamma,$$

[5pts]

implies

$$-m_e^2 = p_e'^2 = (p_e + p_\gamma - p'_\gamma)^2 = m_e^2 + 2p_e \cdot p_\gamma - 2p_e \cdot p'_\gamma - 2p_\gamma \cdot p'_\gamma$$

[5pts]

Now in the lab frame,

$$p_e = (m_e, 0), \quad p_\gamma = (h/\lambda, h/\lambda \hat{e}_0), \quad p'_\gamma = (h/\lambda', h/\lambda' \hat{e}'_0),$$

where \hat{e}_0 is the unit vector of the initial direction and \hat{e}'_0 is the unit vector of the final direction. We get the required result.

[5pts]

- (b) Show that it is impossible for an isolated free electron to absorb or emit a photon.

Soln: We should have the 4-momentum conservation

$$p_e + p_\gamma = p'_e.$$

[5pts]

Squaring both sides implies

$$2p_e \cdot p_\gamma - m_e^2 = -m_e^2$$

i.e.

$$p_e \cdot p_\gamma = 0.$$

In the lab frame,

$$p_e = (m_e, 0), \quad p_\gamma = (E, \mathbf{p}).$$

This implies $E = 0$, i.e. no photon. So the process cannot occur.

[5pts]

4. (Relativistic Fluid)

- (i) For a fluid with mass density ρ , pressure P , temperature T , entropy S and number density n . Derive the first law of thermodynamics.

Soln:

. first law $dE = dQ - pdV$

[2pts]

. derivation (see lecture note p.4.11 for the derivation)

[5pts]

. write down equation

$$nT d\sigma = d\rho - (p + \rho)dn/n$$

[3pts]

- (ii) Write down the energy-momentum tensor for a perfect fluid.

Soln:

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p + \rho)u^\mu u^\nu.$$

[5pts]

- (iii) Use the conservation of energy-momentum

$$\partial_\mu T^{\mu\nu} = 0$$

to show that the flow of a perfect fluid is adiabatic, i.e.

$$\frac{d\sigma}{d\tau} = 0,$$

where τ is the proper time for a fluid element and σ is the entropy per particle.

Soln: We have the conservation of energy-momentum:

$$\frac{\partial p}{\partial x^\alpha} + \frac{\partial}{\partial x^\beta} ((p + \rho)u^\alpha u^\beta) = 0.$$

[2pts]

and conservation of particle number:

$$\frac{\partial}{\partial x^\alpha} (nu^\alpha) = 0.$$

[2pts]

Contract the first eqn with u_α and use

$$u_\alpha \partial_\beta u^\alpha = 0,$$

and the conservation of particle number, we get the desired result.

[6pts]