Chong-Sun Chu

Solution
Mid Examination
Introduction to Relativity I (PHYS431000)

1. (Relativistic Scattering)

(a) Consider a particle of mass m with motion described by $x^i = x^i(t)$, where x^i, $i = 1, 2, 3$ are the spatial coordinates and t is the coordinate time. What is the 4-velocity u^μ of the particle?
Soln: 4-velocity is defined by

$$u^\mu = \frac{dx^\mu}{d\tau}$$

where $d\tau^2 = dt^2 - dx_i^2 = dt^2(1 - v^2)$. *Hence*

$$u^\mu = \gamma(1, v^i)$$

[5pts]

(b) Consider a system of particles with rest masses m_a and velocities $\mathbf{v_a}$, $a = 1, \cdots, N$. Write down the 4-momentum of each particles. State the conservation of energy and momentum.
Soln:

$$P_a^\mu = m_a u_a^\mu$$

$$(\sum_{a=1}^N P_a^\mu)_{\text{initial}} = (\sum_{a=1}^N P_a^\mu)_{\text{final}}$$

[5pts]

(c) Consider the collision of two particles with masses m_1 and m_2. Suppose a single particle of mass M is created. Show that the enegy of the particles are completey fixed. What is the energy of the particle 1? *(Hint: Consider the rest frame of the particle M)*
Soln: Consider rest frame of M, we have the conservation laws:

$$E_1 + E_2 = M, \quad p_1^i = p_2^i.$$

[5pts]

Squaring the first eqn, we obtain

$$E_2^2 = (M - E_1)^2 = M^2 - 2ME_1 + E_1^2.$$

Now, $E_2^2 = p_2^2 + m_2^2$, $E_1^2 = p_1^2 + m_1^2$. *Substitute them and use the momentum conservation law* $p_1^2 = p_2^2$, *we obtain*

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}.$$

It is fixed by the masses. *[10 pts]*

2. (Energy Momentum tensor)

(a) Suppose we use the energy momentum $T^{\mu\nu}$. Ignore gravity, state the conservation of energy-momentum in terms of a conserved equation using $T^{\mu\nu}$. Explain how the 4-momentum P^μ of the system can be defined. Show that P^μ is a conserved quantity.
Soln: Conservation law is

$$\partial_\mu T^{\mu\nu} = 0.$$

$$P^\mu := \int d^3x T^{\mu 0}.$$

It is

$$\begin{aligned}\frac{d}{dt}P^\mu &= \int d^3x \partial_0 T^{\mu 0} \\ &= -\int d^3x \partial_i T^{i0} \\ &= 0\end{aligned}$$

after integrating out the total derivative and assuming the boundary contributions are zero as the system is closed. *[5pts]*

(b) Show that the energy-momentum for a closed system obeys the virial theorem

$$\int T^{ij} d^3x = \frac{1}{2}\frac{d^2}{dt^2}\int T^{00} x^i x^j d^3x.$$

Soln: Using the conservation law, we have

$$\begin{aligned}\frac{d^2}{dt^2}\int T^{00} x^i x^j d^3x &= -\frac{d}{dt}\int \partial_k T^{k0} x^i x^j d^3x \\ &= \frac{d}{dt}\int (T^{i0}x^j + T^{j0}x^i) d^3x \\ &= -\int (\partial_k T^{ik} x^j + \partial_k T^{jk} x^i) d^3x \\ &= 2\int T^{ij} d^3x,\end{aligned}$$

where we have used integration by parts and drops the boundary terms. *[10pts]*

(c) One may model the universe as a collection of dust with a rest mass density $\rho(x^i, t)$ (as measured in their comoving frame) and a local velocity $\mathbf{v}(x^i, t)$. Derive the energy momentum tensor for this universe.

Soln: In the MCRF, we only has rest energy and the energy momentum tensor is

$$\tilde{T}^{00} = \rho, \quad \tilde{T}^{ij} = \tilde{T}^{0i} = 0.$$

Now go to the laboratory frame with a boost,

$$x^\alpha = \Lambda^\alpha{}_\beta(v)\tilde{x}^\beta,$$

we have

$$T^{00} = \rho\gamma^2, \quad T^{i0} = \rho\gamma^2 v^i, \quad T^{ij} = \rho\gamma^2 v^i v^j,$$

or

$$T^{\mu\nu} = \rho u^\mu u^\nu.$$

[10pts]

3. (Covariant derivatives)

(a) State the Einstein version of equivalence principle and the relativity.

Soln: At every spacetime point in an arbitrary gravitational field, it is possible to choose a local inertial coordination system such that within a sufficieintly small region, the law of nature takes the same from as in an unaccelerated Cartesian coordinate system in the absence of gravity. *[5pts]*

(b) Under a general coordinate transformation $x^\mu \to x'^\mu$. How should a tensor $T^{\mu\nu}{}_\alpha$ transform? Partial derivative of tensor does not transform covariantly. To remedy this, we promote it to become a covariant derivative. Write down $T^\mu{}_{\nu;\alpha}$.

Soln:

$$T^{\mu\nu}{}_{\alpha} \to \frac{\partial x'^{\mu}}{\partial x^a}\frac{\partial x'^{\nu}}{\partial x^b}\frac{\partial x^c}{\partial x'^{\alpha}}T^{ab}{}_c$$

[2pts].

It is

$$T^{\mu}{}_{\nu;\alpha} = T^{\mu}{}_{\nu,\alpha} + \Gamma^{\mu}_{\alpha\beta}T^{\beta}{}_{\nu} - \Gamma^{\beta}_{\nu\alpha}T^{\mu}{}_{\beta}.$$

[3pts]

(c) Explain why it follows from the Einstein equivalence principle that

$$g_{\mu\nu;\alpha} = 0,$$

Soln: If we go to a local inertial frame, then the metric is given by the constant metric $\eta_{\mu\nu}$. This implies the connection is zero, and

$$\eta_{\mu\nu;\alpha} = \eta_{\mu\nu,\alpha} = 0$$

Hence $g_{\mu\nu;\alpha} = 0$. *[5pts]*

(d) For an antisymmetric $T^{\mu\nu} = -T^{\nu\mu}$. Derive that

$$T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}T^{\mu\nu}),$$

where $g = |\det g|$ is the determinant of the metric $g_{\mu\nu}$.

Soln:

$$T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda}$$

The last term vanishes due to antisymmetry of T. Now

$$\begin{aligned}\Gamma^{\mu}{}_{\mu\lambda} &= \frac{1}{2}g^{\mu\rho}(\frac{\partial g_{\rho\mu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho}) = \frac{1}{2}g^{\mu\rho}\frac{\partial g_{\rho\mu}}{\partial x^\lambda} \\ &= \frac{1}{2}\frac{\partial}{\partial x^\lambda}\log(-g) \\ &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\sqrt{-g}.\end{aligned}$$

Hence the result. *[10pts]*

4. (Geodesic equation)

(a) Consider a free falling particle in an arbitrary gravitational field. Let ξ^α denotes the local inertial frame and τ the proper time. Show that the equation of motion takes the form

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

in an arbitrary frame x^μ. Here

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{\partial x^\lambda}{\partial\xi^\alpha}$$

is called the affine connection.

Soln: It is known that the equation of motion for a free falling particle is given by

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0,$$

where ξ^a denotes the coordinates of the free falling frame and τ is the proper time. Using

$$\frac{d\xi^\alpha}{d\tau} = \frac{\partial\xi^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau}$$

and chain rule, we arrive at the desired result. *[10pts]*

(b) *Derive the transformation law of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ under a general coordinate transformation $x^{\mu} \to x'^{\mu}$.*

Soln: Under a coordinate transformation $x^{\mu} \to x'^{\mu}$, we have

$$
\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right) \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \Big[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} \Big]
\end{aligned}
$$

[10pts]

(c) *Is affine connection a tensor? Explain.*

Soln: A tensor should transform homogeneously. Since the connection does not transform homogeneously, it is not a tensor. *[5pts]*