Special relativity

- Galilean transf.

For two reference frames K & K' with coordinates (x, y, z, t) and (x', y', z', t'), moving with relative velocity $\vec{v}$, the coordinates are related according to the Galilean relativity.

$$\begin{cases} \vec{x}' = \vec{x} - \vec{v}t \\ t' = t \end{cases}$$

- Classical mechanics are inv under Galilean transf.

For example, in frame K', Newton's law take the form

$$m_i \frac{d\vec{V}_i'}{dt} = -\nabla_i' \sum_j V_{ij}(|\vec{x}_i' - \vec{x}_j'|)$$

for a system of N particles.

In frame K, using $\vec{V}_i' = \vec{V}_i - \underset{\text{constant}}{\vec{v}}$

$$\vec{x}_i' - \vec{x}_j' = \vec{x}_i - \vec{x}_j$$
$$\nabla_i' = \nabla_i$$

we have $m_i \frac{d\vec{V}_i}{dt} = -\nabla_i \sum_j V_{ij}(|\vec{x}_i - \vec{x}_j|)$

The preservation of the <u>form</u> of the eqn of classical mechanics under G.T. is called the principle of <u>Galilean relativity</u>.

Einstein (Special) relativity

Consider the wave equation:

$$\left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\psi = 0 \qquad \text{in frame } K'$$

In ref. frame K, using Galilean transformation, we have

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\vec{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}(v\cdot\nabla)^2\right]\psi = 0$$

The form of the wave eqn. is not inv under Galilean transf.

This appeared for electromagnetism, as well as sound wave.
The vital difference is that for sound waves, the existence of preferred ref. frame can be well understood in terms of the bulk motion of the media of propagation.
For EM, we can also assumed that there is a medium called ether that support the propagation of EM wave.
However, there is no other manifestation of ether other than to support propagation.

When Einstein began to think about these matters, there exists several possibilities:

① . Maxwell eqn were incorrect. The proper theory of EM is inv. under Galilean transf.

② Galilean relativity applied to CM. But EM had a preferred ref. frame, the frame in which the luminiferous ether was at rest.

③ There exists a relativity principle, for both CM & EM, but it was not Galilean relativity. This would imply the law of mechanics were in need of modification.

- The first possibility was hardly viable. Amazing success of the Maxwell theory in the hands of Hertz, Lorentz and others.

The second possibility was accepted by most physicists of the time. Efforts to observe motion of earth relative to the rest frame of ether, e.g. the Michelson-Morley expt., had failed. But the null result could be explained by the FitzGerald-Lorentz contraction hypothesis (1892) whereby objects moving at vel $\vec{v}$ thr. the ether are contracted in the direction of motion according to the formula:

$$L = L_0\sqrt{1 - v^2/c^2}$$

Einstein chose the third alternative. Einstein theory of relativity is based on 2 postulates

1° Postulate of relativity

The laws of nature and results of all experiment performed in a given frame of ref. are indep. of the motion of the system as a whole.

More precisely, there exists a infinite set of equivalent ref. frames moving with constant vel. relative to each other in which all physical phenomena occur in an identical manner.

These equiv. coord system are called inertial ref. frames.

2° Postulate of the constancy of the speed of light.

The speed of light is indep of the motion of its source.

Some Consequences: 1° Modification of law of mechanics at high-speed.

$$E = p^2/2m \rightsquigarrow E^2 = p^2c^2 + m^2c^4$$

Famous $E = mc^2$
rest energy.

2.° Galilean transf $\rightsquigarrow$ Lorentz transf. Space, time symmetrical!

frame K' moving in x-direction wrt frame K

$x_0 = ct$

$$\begin{cases} x_0' = \gamma(x_0 - \beta x_1) \\ x_1' = \gamma(x_1 - \beta x_0) \\ x_2' = x_2 \\ x_3' = x_3 \end{cases}$$

$$0 \le \beta = v/c \le 1$$

$$1 \le \gamma = 1/\sqrt{1 - v^2/c^2} < \infty$$

Inverse Lorentz transf.

$$\begin{cases} x_0 = \gamma(x_0' + \beta x_1') \\ x_1 = \gamma(x_1' + \beta x_0') \\ x_2 = x_2' \\ x_3 = x_3' \end{cases}$$

$$c^2t^2 - \vec{x}^2 = c^2t'^2 - \vec{x}'^2 = \text{inv.}$$

Matrix representation of Lorentz transformation

- Lorentz transf. leave inv.

$$s^2 = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2$$

The transformations form a group called Lorentz group.

The first postulate : Mathematical eqns expressing the laws of nature must be covariant i.e. invariant in form under the transformation of the Lorentz gp.

- Tensors : A rank k tensor associated with a spacetime pt. x is defined by their transf. properties under the transf. $x \to x'$.

A scalar is a single quantity whose value is not changed by the transformation. eg. s^2

A vector is a rank one tensor

Contravariant : $A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}$

Covariant : $B'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} B_{\beta}$

- The norm or metric , $ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$

$g_{\beta\alpha} = g_{\alpha\beta}$ is called the metric tensor.

$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1$

$g^{\alpha\beta}$ = inverse of $g_{\alpha\beta}$. In flat spacetime, $g^{\alpha\beta} = g_{\alpha\beta}$

$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$

Indices are raised & lowered by $g_{\alpha\beta}$ or $g^{\alpha\beta}$.

- Convenient to represent the contravariant 4-vector using a matrix notation:

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$gx = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} \qquad \text{norm} = (x, gx) \qquad (x,y) \triangleq x^T y$$

Linear transformation : $x' = Ax$ $\quad$ A = 4×4 square matrix $\quad$ $x'^\mu = A^\mu{}_\nu \, x^\nu$

Requiring norm inv. $\Rightarrow$ $x'^\mu g_{\mu\nu} x'^\nu = x^\mu g_{\mu\nu} x^\nu$

$$\overset{\parallel}{x^\lambda A^\mu{}_\lambda \, g_{\mu\nu} A^\nu{}_\sigma x^\sigma} \quad \Rightarrow \quad \boxed{g_{\mu\nu} A^\mu{}_\lambda A^\nu{}_\sigma = g_{\lambda\sigma}}$$

Taking det $\Rightarrow$ $(\det A)^2 \det g = \det g$ $\quad \Rightarrow \det A = \pm 1$

$$\begin{cases} \det A = +1 & \text{proper Lorentz transf} \\ \det A = -1 & \text{improper Lorentz transf.} \quad \text{e.g. } A = -I \quad \text{space-time} \end{cases}$$

- Infinitesimal representation $\qquad$ 12 $\qquad$ 13/11

Sometime we want to consider transformation of coordinates that is not "too much":

Representing the matrix A as $A = e^L \triangleq 1 + L + \frac{L^2}{2!} + \cdots$

then the meaning of A being infinitesimal means we keep only the first nontrivial term:

$$A = 1 + L \qquad\qquad A = e^{\epsilon L} = 1 + \epsilon L + o(\epsilon^2)$$

Useful : $\det A = e^{\operatorname{Tr} L}$

$\therefore \det A = 1 \quad \Rightarrow \quad \boxed{\operatorname{Tr} L = 0} \qquad L$ = traceless.

Also, $\underbrace{gA^T g}_{\overset{\parallel}{e^{gL^T g}}} = \underset{\overset{\parallel}{e^{-L}}}{A^{-1}} \quad \Rightarrow \quad \boxed{gL^T g = -L}$

i.e. $(gL)^T = -gL$, gL is antisymmetric.

6 Fundamental matrices ; $\frac{4\times 3}{2} = 6$

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{L} = \underbrace{-\vec{\omega}\cdot\vec{S}}_{\text{rotation}} \; \underbrace{-\vec{\zeta}\cdot\vec{K}}_{\text{Boost}}$$

$\vec{\omega}, \vec{\zeta}$ constant 3-vectors

e.g. $\vec{\omega} = 0, \quad \vec{\zeta} = \zeta\,\vec{e}_1,$

$$A = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \text{Boost.}$$

e.g. $\vec{\omega} = \omega\,\hat{e}_3, \quad \vec{\zeta} = 0$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The six matrices are a reps. of the infinitesimal generators of the Lorentz gp.
It is easy to check that:

$$[S_i, S_j] = \epsilon_{ijk} S_k$$
$$[S_i, K_j] = \epsilon_{ijk} K_k$$
$$[K_i, K_j] = -\epsilon_{ijk} S_k$$

Lorentz gp. $SL(2,\mathbb{C})$ or $SO(3,1)$

- This is isomorphic to the Lie alg. $SL(2,\mathbb{C})$

Pf. Define $L = S + iK$

$R = S - iK$

$$[L,L] = \underbrace{[S,S] - [K,K]}_{2\epsilon S} + i[S,K] \underbrace{- i[K,S]}_{i[S,K]}$$

$$= 2\epsilon L$$

$$[R,R] = 2\epsilon R$$

$$[L,R] = [S,S] + [K,K] - i[S,K] + i[K,S] = 0$$

$\therefore$ L form a Lie alg $SL(2,\mathbb{C})$ } Cpx conjugate to each other.
R .... $SL(2,\mathbb{C})$

$SO(3,1) \simeq SL(2,\mathbb{C})$

Fundamental Principle of QM. (nonrelativistic)

1° Wave function $\psi(q, s, t)$

classical degree of freedom; any additional d.o.f.

$\psi(q,s,t)$ has no direct physical meaning.

$|\psi|^2$ is interpretated as the probability for the system to take values (q,s) at time t.

2° Any observables can be represented by a Hermitian operator acting on ψ.

In particular, canonical momentum

$$P_i = -i\hbar \frac{\partial}{\partial q_i}$$

3° A system is in an eigenstate of an operator Ω if

$$\Omega \phi_n = \omega_n \phi_n$$

For Hermitian op., ω_n is real.

4° Let (Ω_n) be a set of complete set of commuting observables (CSCO), and let ψ_n be the complete set of orthonormal wavefunctions. Then any state ψ can be written as

$$\psi = \sum_n a_n \psi_n$$

Completeness relation is $\sum_s \int dq \; \psi_n^*(q,s,t)\, \psi_m(q,s,t) = \delta_{nm}$

$|a_n|^2$ is the prob in the state ψ_n

5° The observed value in any observation must be the eigenvalue of that operator.

In particular, if we have a system described by a state $\psi = \sum a_n \psi_n$, then the average observed value for the observable Ω is

$$\langle \Omega \rangle_\psi = \sum_s \int dq \; \psi^* \, \Omega \, \psi$$

6° The time evolution of a physical system is governed by the Schrodinger eqn.

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

Hermitian of H implies Conservation of probability:

$$\frac{d}{dt} \sum_s \int dq \; \psi^* \psi = 0$$

We will try to keep these fundamental principle when we generalize to relativistic QM.

Klein Gordon eqn

Let us review the Schrodinger eqn and how one "derive" it.

Hamiltonian for a free isolated particle

$$H = p^2/2m \qquad \text{(non relativistic)}$$

Use the substitution $H \to i\hbar \frac{\partial}{\partial t}$

$p \to -i\hbar \nabla$

This leads to the Schrodinger eqn.

$$i\hbar \frac{\partial}{\partial t}\psi = \frac{-\hbar^2 \nabla^2}{2m}\psi \qquad \text{(non relativistic)}$$

This eqn. is nonrelativistic and is not covariant

↑ i.e. LHS and RHS transforms differently under Lorentz transformation

Relativistically, four vector $p^\mu = (p^0, p^1, p^2, p^3) = (E/c, p_x, p_y, p_z)$

has invariant length

$$\sum_{\mu=0}^{3} p_\mu p^\mu = E^2/c^2 - \vec{p}\cdot\vec{p} = m^2c^2$$

↖ rest mass of particle

Naturally take Hamiltonian to be

$$H = \sqrt{p^2c^2 + m^2c^4}$$

and try the generalization

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}\;\psi$$

Problems: ① How to interpret the $\sqrt{\ }$?

if we do an expansion, then we get

$$\sqrt{-\hbar^2c^2\nabla^2 + m^2c^4} = mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\nabla^2} = mc^2\left(1 - \frac{\hbar^2}{2m^2c^2}\nabla^2 + \nabla^4 + \cdots\right)$$

infinite power of ∇^2 ⇒ highly nonlocal!

② also, space and time appears asymmetrically!

For reason of mathematical symplicity, we can also try

$H^2 = p^2c^2 + m^2c^4$

in fact, if $[A,B]=0$, then $A\psi = B\psi$

$\Rightarrow A^2\psi = B^2\psi$

Now $A = i\hbar\frac{\partial}{\partial t}$

$B = \sqrt{\quad}$ = space-derivatives

leads to $-\hbar^2\frac{\partial^2}{\partial t^2}\psi = (-\hbar^2\nabla^2c^2 + m^2c^4)\psi$

i.e. $\boxed{\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0}$ Klein-Golden eqn — ①

$\Box = \partial_\mu\partial^\mu = \frac{1}{c^2}\partial_t^2 - \nabla^2$ $\partial_\mu = (\frac{1}{c}\partial_t, \partial_x)$ $\partial^\mu = (\frac{1}{c}\partial_t, -\partial_x)$

- Remark: Since we square the operator, we have introduced a negative root

$H = -\sqrt{p^2c^2 + m^2c^4}$

i.e. ① describes in principle also soln with $E<0$!

What is the meaning of negative energy soln ?

will see this is related to anti-particle! → Dirac

- Probability interpretation and conservation of probability

Recall, Schrodinger eqn. $i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$

$$\underbrace{\psi^* i\hbar\frac{\partial}{\partial t}\psi - \psi\left(-i\hbar\frac{\partial}{\partial t}\psi^*\right)}_{i\hbar\frac{\partial}{\partial t}(\psi^*\psi)} = \underbrace{-\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)}_{-\frac{\hbar^2}{2m}\nabla\cdot[\psi^*\nabla\psi - (\nabla\psi^*)\psi]}$$

$\rho = \psi^*\psi$, $\vec{j} = \frac{\hbar}{2im}\left(\psi^*\nabla\psi - (\nabla\psi)^*\psi\right)$

$\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} = 0$

Similarly, $\psi^*\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi - \psi\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi^* = 0$

$$\Rightarrow \nabla^\mu(\psi^* \nabla_\mu \psi - \psi \nabla_\mu \psi^*) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\underbrace{\left[\frac{i\hbar}{2mc^2}(\psi^* \partial_t \psi - \partial_t \psi^* \cdot \psi)\right]}_{?\ \rho\ ?} + \nabla \cdot \underbrace{\left[\frac{\hbar}{2im}(\psi^*(\nabla\psi) - (\nabla\psi)^*\psi)\right]}_{\vec{J}} = 0$$

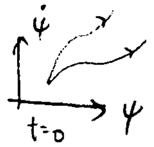
But this is not good as ρ is not positive definite!

pf. ρ depends on ψ & $\partial_t \psi$

KG eqn is second order PDE, need to specify

$\psi(\vec{x},0)$ & $\frac{\partial}{\partial t}\psi(\vec{x},0)$

But these are completely arbitrary and so $\psi(\vec{x},t)$, $\partial_t \psi(\vec{x},t)$ can be quite arbitrary too.



$\therefore$ ρ does not need to be positive definite!

Dirac eqn.

Dirac introduce the Dirac eqn in 1928, he tried to keep:

1) positivity of probability : $\rho \geq 0$

2) Conservation of probability : $\frac{d}{dt}\int \rho \, d^3x = 0$

3) Lorentz inv.

Pauli introduced 2 components wave function in his QM description of spin of electron. (nonrel.)
Dirac took a similar step, he introduced a multicomponent wave fun:

$$\psi = \begin{pmatrix} \psi_1(\vec{r},t) \\ \psi_2(\vec{r},t) \\ \vdots \\ \psi_N(\vec{r},t) \end{pmatrix}$$

He defines probability $\rho = \sum_{\nu=1}^{N} \psi_\nu^* \psi_\nu = \psi^+ \psi$ ← positive definite

$$\frac{d}{dt}\int \rho \, d^3x = \sum_\nu \int \frac{\partial \psi_\nu^*}{\partial t}\psi_\nu + \psi_\nu^* \frac{\partial \psi_\nu}{\partial t} \, d^3x \overset{!}{=} 0$$

To satisfy this eqn, $\frac{\partial}{\partial t}\psi$ can't be given arb. ⇒ wave eqn should depends only in 1st order derivative in time!

Relativistic inv. ⇒ equal status of $\vec{x}$ & t ⇒ also 1st derivative in $\vec{x}$

∴ Dirac suggested the eqn for free electron to take the form:

$$i\hbar\frac{\partial}{\partial t}\psi_\sigma = \frac{\hbar c}{i}\sum_{\tau=1}^{N}\left(\alpha_1\frac{\partial}{\partial x_1} + \alpha_2\frac{\partial}{\partial x_2} + \alpha_3\frac{\partial}{\partial x_3}\right)_{\sigma\tau}\psi_\tau + \sum_{\tau=1}^{N}\beta_{\sigma\tau}\psi_\tau \cdot mc^2 = \sum_{\tau=1}^{N} H_{\sigma\tau}\psi_\tau$$

i.e. $i\hbar\frac{\partial}{\partial t}\psi = \frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla}\psi + mc^2\beta\psi$

$$H = \frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla} + mc^2\beta = c\vec{\alpha}\cdot\vec{p} + mc^2\beta$$

Properties of $\vec{\alpha}$, β are to be determined.

- H Hermitian $\Rightarrow$ $\alpha^+ = \alpha$, $\beta^+ = \beta$

- Probability conservation: $\frac{1}{c}\frac{\partial}{\partial t}(\psi^+\psi) + \psi^+\vec{\alpha}\cdot\vec{\nabla}\psi + (\nabla\psi^+)\cdot\vec{\alpha}\,\psi = 0$

  let $\rho = \psi^+\psi$, $\vec{j} = c\psi^+\vec{\alpha}\psi$

  then $\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} = 0$

- We also require each component of ψ to satisfy the K-G eqn., (Einstein's relation $E^2 = m^2c^4 + p^2c^2$,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$$

We have
$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi = -\hbar^2c^2\sum\underbrace{\frac{\alpha_i\alpha_j + \alpha_j\alpha_i}{2}}_{\overset{!}{=}\,\delta_{ij}}\frac{\partial^2\psi}{\partial x^i\partial x^j} + \frac{\hbar mc^3}{i}\sum\underbrace{(\alpha_i\beta + \beta\alpha_i)}_{\overset{!}{=}\,0}\partial_i\psi + \underbrace{\beta^2}_{\overset{!}{=}\,1}m^2c^4\psi$$

Need
$$\begin{cases}\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}\\ \alpha_i\beta + \beta\alpha_i = 0\\ \beta^2 = 1\end{cases}$$

ie
$$\begin{cases}\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1\\ \alpha_1, \alpha_2, \alpha_3, \beta \quad \text{anticommute with each other.}\end{cases}$$

<u>Representation</u>

$\because\ \alpha_i = -\beta\alpha_i\beta$

$\Rightarrow \mathrm{Tr}\,\alpha_i = -\mathrm{Tr}\,\alpha_i = 0$

$\because\ \alpha_i^2 = 1 \Rightarrow$ eigenvalues $= \pm 1$

$\Bigg] \Rightarrow \alpha_i, \beta$ even dimensional !

The min dim. is 2. It is excluded since we can only have 3 Pauli matrix and the unit matrix
not traceless !

The next dim is 4. An explicit representation is

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Pauli representation.

<u>Ex</u> verify it

γ^μ

- More often write Dirac eqn in a form that space, time are symmetrical:

Multiply $i\hbar \frac{\partial}{\partial t}\psi = \frac{\hbar c}{i}\vec{\alpha}\cdot\vec{\nabla}\psi + mc^2 \beta\psi$ by β/c

$$\underbrace{i\hbar\left[\beta\frac{1}{c}\frac{\partial}{\partial t} + \beta\vec{\alpha}\cdot\vec{\nabla}\right]}_{\gamma^\mu\partial_\mu}\psi - mc\psi = 0$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\right)$$

$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i$

$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$

or $(\not{p} - mc)\psi = 0$, $\not{p} = \gamma^\mu p_\mu$

$p_\mu = +i\hbar\partial_\mu$

$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = +2g^{\mu\nu}\mathbb{1}$

verify: $(\gamma^0)^2 = 1$

$(\gamma^i)^2 = \beta\alpha^i\beta\alpha^i = -\beta^2\alpha^{i2} = -1$ //

$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$

γ^μ are called Gamma matrices.

In Dirac-Pauli reps, $\gamma^i = \begin{pmatrix}0 & \sigma_i \\ -\sigma_i & 0\end{pmatrix}$, $\gamma^0 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$

- <u>Reps. independence</u>

Suppose somebody writes

$$\left(i\gamma'_\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi' = 0$$

only defining property of γ'_μ is used: $\{\gamma'_\mu, \gamma'_\nu\} = +2g_{\mu\nu}$

claim. This eqn is the same as the original Dirac eqn.

Need: Pauli Fund. Thm: Given two set of matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} = +2g_{\mu\nu} = \{\gamma'_\mu, \gamma'_\nu\}$$

there exist a non singular 4×4 matrix such that

$$S\gamma_\mu S^{-1} = \gamma'_\mu$$

Moreover S is unique up to a multiplicative constant.

pf eg. Sakurai AQM appendix C.

Thus, we can write,

$$\left(i S \gamma_\mu S^{-1} \partial^\mu - \frac{mc}{\hbar}\right) S S^{-1} \psi' = 0$$

$$\Rightarrow S\left(i\gamma_\mu \partial^\mu - \frac{mc}{\hbar}\right) S^{-1}\psi' = 0$$

ie $\left(i\gamma_\mu \partial^\mu - \frac{mc}{\hbar}\right) S^{-1}\psi' = 0$

This is the same as the original Dirac eqn. with $S^{-1}\psi'$ as soln.

ie. The two eqn. are equivalent, with the wave fn ψ & ψ' related by

$$\psi' = S\psi$$

If γ^μ is Hermitian, then

$$(S\gamma_\mu)^+ = (\gamma_\mu' S)^+ = S^+ \gamma_\mu' = S^+ S \gamma_\mu S^{-1}$$

$$\gamma_\mu S^+$$

$$\Rightarrow [\gamma_\mu, S S^+] = 0$$

Can take $S^+ = S^{-1}$ unitary ..

4-current

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{J} = 0 \qquad \rho = \psi^+\psi, \quad \vec{J} = c\psi^+\vec{\alpha}\psi$$

want to write it as $\partial_\mu j^\mu = 0$

$$= \frac{\partial}{c\partial t} j^0 + \nabla\cdot\vec{j} = 0$$

Define $j^0 = \psi^+\psi$, $j^i = \psi^+\alpha^i\psi$

Introduce $\bar{\psi} = \psi^+\gamma^0$

then $j^\mu = \bar{\psi}\gamma^\mu\psi$ //

Note that j^μ is reps. indep.

$$\bar{\psi}'\gamma'^\mu\psi' = \psi'^+\gamma'^0\gamma'^\mu\psi' = \psi' \underbrace{S^+S}_{1} \gamma^0\gamma^\mu \underbrace{S^+S}_{1} \psi = \bar{\psi}\gamma^\mu\psi$$

//

Nonrelativistic limit

- This help us to see better the physical meaning of Dirac eqn.

Consider a free electron at rest.

$$i\hbar \frac{\partial}{\partial t}\psi = \beta mc^2 \psi \qquad \beta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

4 soln. $\psi^1 = e^{-imc^2/\hbar\, t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\psi^2 = e^{-imc^2/\hbar\, t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\psi^3 = e^{imc^2/\hbar\, t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $\psi^4 = e^{imc^2/\hbar\, t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

ψ^1, ψ^2 corresponds to positive energy, ψ^3, ψ^4 correspond to $-$ve energy.

? Need to be understood

For the moment, we want to show that the positive energy soln. has a meaningful NR simplification.

- Consider coupling to EM field, $p^\mu \to p^\mu - \frac{e}{c}A^\mu \qquad A^\mu = (\Phi, \vec{A})$

Dirac eqn. becomes $i\hbar \frac{\partial}{\partial t}\psi = \left(c\,\vec{\alpha}\cdot(\vec{p} - \frac{e}{c}\vec{A}) + \beta mc^2 + e\Phi\right)\psi$

$$H = H_0 + H' \qquad H_0 = c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2$$
$$H' = -\,\vec{\alpha}\cdot\vec{A} + e\,\Phi$$

Compared to classical interaction energy $H'_{classical} = -\frac{e}{c}\vec{v}\cdot\vec{A} + e\Phi$

we see that the matrice $c\vec{\alpha}$ represent vel. in Dirac eqn.

· To take NR limit, it is convenient to represent ψ as

$$\psi = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} \begin{matrix} \leftarrow \text{2 components} \\ \leftarrow \text{wave fun.} \end{matrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then $$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} = c\, \vec{\sigma}\cdot\vec{\pi} \begin{pmatrix} \tilde{\chi} \\ \tilde{\varphi} \end{pmatrix} + e\Phi \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} + mc^2 \begin{pmatrix} \tilde{\varphi} \\ -\tilde{\chi} \end{pmatrix}, \quad \vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$$

In NR limit, rest energy mc^2 is the largest part of energy. If we write

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} = e^{-imc^2 t/\hbar} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

Then φ, χ is a slow varying function of time.

$$\rightarrow \quad i\hbar \frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c\vec{\sigma}\cdot\vec{\pi}\begin{pmatrix} \chi \\ \varphi \end{pmatrix} + e\Phi \begin{pmatrix} \varphi \\ \chi \end{pmatrix} - 2mc^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

$mc^2 \gg$ K.E. + interaction $e\Phi$

$$\therefore \quad \chi \simeq \frac{\vec{\sigma}\cdot\vec{\pi}}{2mc}\varphi$$

Sub into 1st component,

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})}{2m} + e\Phi \right] \varphi$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \pi^2 + i\vec{\sigma}\cdot(\vec{\pi}\times\vec{\pi})$$

$$\vec{\pi}\times\vec{\pi} = \left(\vec{p} - \frac{e}{c}\vec{A}\right)\times\left(\vec{p} - \frac{e}{c}\vec{A}\right) = -\frac{e}{c}\left(\vec{p}\times\vec{A} + \vec{A}\times\vec{p}\right)$$

$$= \frac{i\hbar e}{c}\left(\vec{\nabla}\times\vec{A} + A\times\vec{\nabla}\right) \underset{\uparrow}{=} \frac{i\hbar e}{c}\vec{B}$$

$$(\nabla\times\vec{A} + \vec{A}\times\nabla) f = \nabla\times(\vec{A}f) + \vec{A}\times(\nabla f) = (\nabla\times\vec{A}) f = \vec{B} f$$

$$\therefore \quad i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + e\Phi \right] \varphi$$

↗ Pauli: 2 component wave eq. for electron !

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \end{matrix} \text{ spin up/down}$$

like Schrodinger eqn.

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Further simplification, we consider uniform magnetic field $\vec{B} = \nabla \times \vec{A}$, $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{\vec{p}^2}{2m} - \frac{e}{2m}\left(\vec{r}\times\vec{p} + \hbar\vec{\sigma}\right)\cdot\vec{B}\right]\varphi + O(B^2)$$

$$= \left[\frac{\vec{p}^2}{2m} - \frac{e}{2m}\left(\vec{L} + 2\vec{S}\right)\cdot\vec{B}\right]\varphi + O(B^2)$$

$$(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p})\varphi$$
$$= p_i(A_i\varphi) + A_i(p_i\varphi)$$
$$= 2A_i(p_i\varphi) \qquad (p_iA_i) = 0$$
$$= \varepsilon_{ijk}B_j r_k (p_i\varphi)$$
$$= \vec{B}\cdot(\vec{r}\times\vec{p})\varphi$$

$\vec{L} = \vec{r}\times\vec{p}$ orbital angular momentum

$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ spin of electron.

interaction of angular momentum and magnetic field takes general form

$$g\frac{e}{2m}\vec{J}\cdot\vec{B}$$

g – g-ratio for $\vec{J}$

for electron, $g_s = 2$!

This is a direct consequence of Dirac eqn!

Big Success of Dirac eqn!

Covariance of Dirac eqn.

In frame K, $(i\hbar \gamma^\mu \partial_\mu - mc)\psi(x) = 0$ ↖ same form — ①

In frame K', $(i\hbar \gamma^\mu \partial'_\mu - mc)\psi'(x') = 0$ ↙ — ②

$x' = Ax$

Question. How is ψ & ψ' related?

We demand that ψ' & ψ are linearly related:

$$\psi'(x') = S(A)\,\psi(x)$$

$$\psi'(x') = \psi'(Ax)$$

ie $\psi(x) = S^{-1}(A)\,\psi'(Ax)$

But also $\psi(x) = S(A^{-1})\,\psi'(Ax)$

$\therefore S^{-1}(A) = S(A^{-1})$

$$S(A) \times ① \Rightarrow \left(i\hbar S(A)\gamma^\mu \partial_\mu - mc\,S(A)\right) S(A^{-1})\psi'(Ax) = 0$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \cdot \frac{\partial}{\partial x'^\nu} = A^\nu{}_\mu\, \partial'_\nu$$

$$\Rightarrow \left[\underbrace{i\hbar S(A)\gamma^\mu S^{-1}(A)\, A^\nu{}_\mu}_{\overset{!}{=}\ \gamma^\nu}\, \partial'_\nu - mc\right]\psi'(x') = 0$$

Want to find $S(A)$ s.t.

$$S(A)\gamma^\mu S^{-1}(A) = (A^{-1})^\mu{}_\nu\, \gamma^\nu$$

or $$\boxed{S(A)^{-1}\gamma^\mu S(A) = A^\mu{}_\nu\, \gamma^\nu} \qquad — ③$$

The problem of demonstrating rel covariance of Dirac eqn is now reduced to that of finding an S that satisfies ③

I. First consider infinitesimal (proper) transf.

$$A^\nu{}_\mu = \delta^\nu_\mu + \omega^\nu{}_\mu \quad , \quad \omega \ll 1$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

let $S = 1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}$, $S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}$

$\sigma_{\mu\nu}$ to be determined. $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$.

Now ③ gives, $\left(1 + \tfrac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\gamma^K\left(1 - \tfrac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\right) = \left(\delta^K{}_\rho + \omega^K{}_\rho\right)\gamma^\rho$

$$\text{L.S.} = \gamma^K + \tfrac{i}{4}\omega^{\mu\nu}[\sigma_{\mu\nu}, \gamma^K] + O(\omega^{\alpha\beta})^2$$

$$\text{RS} = \gamma^K + \omega^K{}_\rho\,\gamma^\rho = \gamma^K + \omega^{K\rho}\gamma_\rho = \gamma^K + \tfrac{1}{2}\omega^{\mu\nu}\left(\delta^K_\mu\gamma_\nu - \delta^K_\nu\gamma_\mu\right)$$

$$\Rightarrow \tfrac{i}{2}[\sigma_{\mu\nu}, \gamma^K] = \delta^K_\mu\gamma_\nu - \delta^K_\nu\gamma_\mu \qquad — ④$$

∴ Our problem reduces to finding the 6 matrices $\sigma_{\mu\nu}$.

Try $\sigma_{\mu\nu} = c\,[\gamma_\mu, \gamma_\nu]$

then $[\sigma_{\mu\nu}, \gamma^K] = c\big[[\gamma_\mu, \gamma_\nu], \gamma^K\big] = c\big(\gamma_\mu\gamma_\nu\gamma^K - \gamma_\nu\underbrace{\gamma_\mu\gamma^K}_{-\gamma^K\gamma_\mu + 2\delta^K_\mu} - \overbrace{\gamma^K\gamma_\mu}^{-\gamma_\mu\gamma^K + 2\delta^K_\mu}\gamma_\nu + \gamma^K\gamma_\nu\gamma_\mu\big)$

$$= c\left[\gamma_\mu\gamma_\nu\gamma^K + \gamma_\nu\gamma^K\gamma_\mu - 2\delta^K_\mu\gamma_\nu + \gamma_\mu\gamma^K\gamma_\nu - 2\delta^K_\mu\gamma_\nu + \gamma^K\gamma_\nu\gamma_\mu\right]$$

$$= c\left(\gamma_\mu 4\delta^K_\nu - 4\gamma_\nu\delta^K_\mu\right)$$

$$= -4c\left(\delta^K_\mu\gamma_\nu - \delta^K_\nu\gamma_\mu\right)$$

$\Rightarrow -2ic = 1$

$c = +i/2$

$$\boxed{S = 1 + \tfrac{1}{8}[\gamma_\mu, \gamma_\nu]\,\omega^{\mu\nu}}$$

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II/ Parity transf. $\vec{x}' = -\vec{x}$
$t' = t$

$$A^{\nu}{}_{\mu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Let $P = S(A)$ be the corresponding soln to ③ ;

$$P^{-1} \gamma^{\mu} P = A^{\mu}{}_{\nu} \gamma^{\nu}$$

Easy to get soln. $\boxed{P = \gamma^0}$

Bilinear Covariants

- Given γ matrices, one can construct 16 4x4 matrices $\Gamma^{n}_{\alpha\beta}$

$$\Gamma^S = 1, \quad \Gamma^V_{\mu} = \gamma_{\mu}, \quad \Gamma^T_{\mu\nu} = \sigma_{\mu\nu}$$
$$\Gamma^P = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5, \quad \gamma^A_{\mu} = \gamma^5 \gamma_{\mu}$$

Easy to show that :

1° $(\Gamma^n)^2 = \pm 1$

2° Except for Γ^S, there is a Γ^m for any Γ^n st

$$\Gamma^n \Gamma^m = -\Gamma^m \Gamma^n$$

Hence $\pm \mathrm{tr}\, \Gamma^n = \mathrm{tr}\, \Gamma^n (\Gamma^m)^2 = -\mathrm{tr}\, \Gamma^m \Gamma^n \Gamma^m = -\mathrm{tr}\, \Gamma^n (\Gamma^m)^2 = \mp \mathrm{tr}\, \Gamma^n = 0$

3° Given Γ^a, Γ^b $(a \neq b)$, $\exists\ \Gamma^n \neq \Gamma^S$ st.

$$\Gamma^n = \Gamma^a \Gamma^b$$

Hence, Γ^n are linear indep.

Pf. let $\sum_{n=1}^{16} a_n \Gamma^n = 0$

multiply $\Gamma^m \neq \Gamma^S$ and take trace $\Rightarrow a_m = 0$ //

Cor Hence any 4x4 matrices can be written as a sum of Γ^n's !

- Covariant bilinear

We can construct $\bar{\psi}(x)\Gamma^n\psi(x) = J^n(x)$

J^n has definite transformation properties under Lorentz transformations.

<u>Claim</u> :

		Proper Lorentz transformation a	Parity P
Scalar	$\bar\psi\psi$	$\bar\psi\psi$	$\bar\psi\psi$
Vector	$\bar\psi\gamma^\mu\psi$	$a^\mu{}_\nu\,\bar\psi\gamma^\nu\psi$	$\begin{cases}-\bar\psi\gamma^k\psi\\ \bar\psi\gamma^0\psi\end{cases}$
Tensor	$\bar\psi\sigma^{\mu\nu}\psi$	$a^\mu{}_\alpha a^\nu{}_\beta\,\bar\psi\sigma^{\alpha\beta}\psi$	$\begin{cases}\bar\psi\sigma^{kl}\psi\\ -\bar\psi\sigma^{k0}\psi\end{cases}$
Axial vector	$\bar\psi\gamma^5\gamma^\mu\psi$	$a^\mu{}_\nu\,\bar\psi\gamma^5\gamma^\nu\psi$	$\begin{cases}\bar\psi\gamma^5\gamma^k\psi\\ -\bar\psi\gamma^5\gamma^0\psi\end{cases}$
Pseudo scalar	$\bar\psi\gamma_5\psi$	$\bar\psi\gamma_5\psi$	$-\bar\psi\gamma_5\psi$

eg. $\bar\psi\gamma^5\gamma^\mu\psi \xrightarrow{P} \psi^+\gamma^0\gamma^5\gamma^\mu\gamma^0\psi = \begin{cases}-\bar\psi\gamma^5\gamma^0\psi\\ \bar\psi\gamma^5\gamma^i\psi\end{cases}$

$-$; $+ : \mu=0$; $- : \mu=i$

- $\bar\psi = \psi^+\gamma^0$

$\psi' = S(\Lambda)\psi$

$\bar\psi' = (\psi')^+\gamma^0 = \psi^+ S(\Lambda)^+\gamma^0$

γ^0 Hermitian $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

γ^i antihermitian $\begin{pmatrix}0&\vec\sigma\\-\vec\sigma&0\end{pmatrix}$

Proper : $S(\Lambda) = 1 + \frac{1}{8}[\gamma^\mu,\gamma^\nu]\,\omega_{\mu\nu}$

For rotation, $\gamma^{ij}\gamma_0 = \gamma_0\gamma^{ij} \Rightarrow (\gamma^{ij})^+\gamma_0 = -\gamma^{ij}\gamma_0 = -\gamma_0\gamma^{ij}$

$\Rightarrow S^+\gamma_0 = \gamma_0 S^{-1}$

For Lorentz transf, $\gamma^{0i}\gamma_0 = -\gamma_0\gamma^{0i} \Rightarrow (\gamma^{0i})^+\gamma_0 = +\gamma^{0i}\gamma_0 = -\gamma_0\gamma^{0i}$

$\Rightarrow S^+\gamma_0 = \gamma_0 S^{-1}$

Always $S^+\gamma_0 = \gamma_0 S^{-1}$

Parity : $P\gamma_0 = \gamma_0\gamma_0 = \gamma_0 S^{-1}$

$\therefore \bar\psi' = \psi^+ S^+\gamma_0 = \psi^+\gamma_0 S^{-1} = \bar\psi S^{-1}$

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Spin angular momentum

$H = c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2$

Obviously $[\vec{p}, H] = 0$ $\Rightarrow$ $\vec{p}, H$ can be taken as observable.

- Velocity operator:

$$\frac{d}{dt}x = \frac{1}{i\hbar}[x, H]$$
$$= \frac{1}{i\hbar}[x, c\,\vec{\alpha}\cdot\vec{p} + \beta m^2]$$
$$= \frac{1}{i\hbar}[x, c\,\alpha_x p_x] = c\,\alpha_x$$

$\therefore$ $\vec{v} = c\,\vec{\alpha}$ velocity operator.

- Orbital angular momentum $\vec{\ell}$

$$\frac{d}{dt}\ell_x = \frac{1}{i\hbar}[\ell_x, H] = \frac{c}{i\hbar}[\ell_x, \alpha_x p_x + \alpha_y p_y + \alpha_z p_z]$$
$$= \frac{c}{i\hbar}\Big\{\alpha_x \underbrace{[\ell_x, p_x]}_{0} + \alpha_y \underbrace{[\ell_x, p_y]}_{i\hbar p_z} + \alpha_z \underbrace{[\ell_x, p_z]}_{-i\hbar p_y}\Big\}$$
$$= c(\alpha_y p_z - \alpha_z p_y)$$
$$\frac{d}{dt}\vec{\ell} = c\,(\vec{\alpha}\times\vec{p})$$

i.e. $[\vec{\ell}, H] = i\hbar c\,(\vec{\alpha}\times\vec{p}) \neq 0$ Cannot take $\vec{\ell}$ to be in the set of commuting observables.

- However space is isotropic

$\Rightarrow$ Expect to see angular momentum to conserve !

$\Rightarrow$ Define $\vec{J} = \vec{\ell} + \vec{s}$ ← spin, intrinsic angular momentum

Q: What form of $\vec{s}$ would allow $\vec{J}$ to be conserved ?

$$[\vec{J}, H] = 0$$

Let's introduce operator $\vec{\Sigma}$ s.t. $\begin{cases} [\vec{\Sigma},\beta]=0, & [\Sigma_i,\alpha_i]=0 \quad i=x,y,z \\ & [\Sigma_i,\alpha_j]=2i\,\varepsilon_{ijk}\,\alpha_k \end{cases}$ — ①

then
$$[\Sigma_i, H] = [\Sigma_i, c\,\vec{\alpha}\cdot\vec{p}]$$
$$= c[\Sigma_i,\alpha_j]\,p_j$$
$$= 2ic\,\varepsilon_{ijk}\,\alpha_k\,p_j$$
$$\Rightarrow [\vec{\Sigma}, H] = -2ic\,\vec{\alpha}\times\vec{p}$$

$\therefore$ let $\vec{S} = \frac{\hbar}{2}\Sigma$

then $[\vec{J}, H] = 0$

Moreover, since $\vec{S}$ should have eigenvalue $\pm\hbar/2$ in any directions,

So requires $\Sigma_x^2 = \Sigma_y^2 = \Sigma_z^2 = 1$ — ②

Also $\vec{S}$ being angular momentum $\Rightarrow$ require $[S_x, S_y] = i\hbar S_z$ etc.

i.e. $[\Sigma_x, \Sigma_y] = 2i\,\Sigma_z$ etc. — ③

①, ②, ③ summarizes the properties of $\vec{\Sigma}$!

In Pauli-Dirac reps., can check that
$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$
satisfies ①, ②, ③ !

Plane wave Soln

- Now let us return to the free-particle problem, this time with $\vec{p} \neq 0$

Substituting $\psi = u(\vec{p})\, e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar}$

into $i\hbar \frac{\partial}{\partial t}\psi = (c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2)\,\psi$

$$E u = (c\vec{\alpha}\cdot\vec{p} + \beta m c^2)\, u$$

let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ $\begin{matrix} \left.\begin{matrix} u_1 \\ u_2 \end{matrix}\right\} \varphi \\ \left.\begin{matrix} u_3 \\ u_4 \end{matrix}\right\} \chi \end{matrix}$

$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

then $\begin{cases} (E - mc^2)\varphi - c\vec{\sigma}\cdot\vec{p}\,\chi = 0 \\ -c\vec{\sigma}\cdot\vec{p}\,\varphi + (E + mc^2)\,\chi = 0 \end{cases}$

Non trivial Soln only if $\begin{vmatrix} E - mc^2 & -c\vec{\sigma}\cdot\vec{p} \\ -c\vec{\sigma}\cdot\vec{p} & E + mc^2 \end{vmatrix} = 0$

ie $E^2 - m^2c^4 = c^2p^2$

ie $E = E_\pm = \pm\sqrt{m^2c^4 + p^2c^2}$

E_+ : positive energy soln.

E_- : negative energy soln.

- We found

$$\begin{cases} \chi = \dfrac{c}{E + mc^2}(\vec{\sigma}\cdot\vec{p})\,\varphi & \text{— (1)} \\[2ex] \varphi = \dfrac{c}{E - mc^2}(\vec{\sigma}\cdot\vec{p})\,\chi & \text{— (2)} \end{cases}$$

But we cannot determine the wave function φ, χ even though $E, \vec{p}$ is given!
This means there are new degree of freedoms, $(H, \vec{p})$ is not a complete set yet

Angular momentum : Spin $\vec{S}$?
orbital $\vec{\ell}$?
Total $\vec{J}$?

Note that $[\vec{J}, \vec{H}] = 0$

But $\vec{J}$ & $\vec{p}$ does not commute. eg. $[J_i, p_j] = [\ell_i, p_j] = i\hbar \varepsilon_{ijk} p_k$

However it is not hard to show that

$$[\vec{\Sigma}\cdot\vec{p}, H] = 0$$

pf. $[\vec{\Sigma}\cdot\vec{p}, H] = [\Sigma_i p_i, H] = \underbrace{[\Sigma_i, H]}_{2iC\varepsilon_{ijk}\alpha_k p_j} p_i + \Sigma_i \underbrace{[p_i, H]}_{0}$

Use $[AB, C] = A[B,C] + [A,C]B$ //

∴ Can take $(\vec{p}, H, \vec{\Sigma}\cdot\vec{p})$ as a CSCO.

Let $(\vec{\Sigma}\cdot\vec{p})u = \lambda u$ ← eigenvalue

Since $(\vec{\Sigma}\cdot\vec{p})^2 = \Sigma_i p_i \Sigma_j p_j = \underbrace{\Sigma_i \Sigma_j}_{i\varepsilon_{ijk}\Sigma_k + \delta_{ij}} p_i p_j = p^2$

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

∴ $\lambda = \pm p$

In Dirac-Pauli reps., $\vec{\Sigma}\cdot\vec{p} = \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}$

∴ $\begin{cases} \vec{\sigma}\cdot\vec{p}\ \varphi = \lambda\varphi & \text{— (3)} \\ \vec{\sigma}\cdot\vec{p}\ \chi = \lambda\chi & \text{— (4)} \end{cases}$

Solve φ (for example):

Write $\varphi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$

∴ For $\lambda = p$, $\dfrac{u_1}{u_2} = \dfrac{p + p_z}{p_x + ip_y} = \dfrac{p_x - ip_y}{p - p_z}$

For $\lambda = -p$, $\dfrac{u_1}{u_2} = -\dfrac{p - p_z}{p_x + ip_y} = -\dfrac{p_x - ip_y}{p + p_z}$

One can then use ①, ② to solve for χ.

Summarizing, we have: Given $\vec{p}$, we have soln $E = E_\pm$, $\vec{\Sigma}\cdot\vec{p} = \pm p$

(a) $E = E_+$, $\vec{\Sigma}\cdot\hat{p} = +1$

$$\psi = \begin{pmatrix} p + p_z \\ p_x + i p_y \\ cp(p+p_z)/(E_+ + mc^2) \\ cp(p_x + i p_y)/(E_+ + mc^2) \end{pmatrix} e^{i(\vec{p}\cdot\vec{x} - E_+ t)/\hbar} \qquad \chi = \frac{c}{E_+ + mc^2}\underbrace{(\vec{\sigma}\cdot\vec{p})\varphi}_{p\varphi}$$

(b) $E = E_+$, $\vec{\Sigma}\cdot\hat{p} = -1$

$$\cancel{\psi = \begin{pmatrix} -(p - p_z) \\ p_x + i p_y \\ \frac{c}{E_+ + mc^2}\, p(p - p_z) \\ \frac{-c}{E_+ + mc^2}\, p(p_x + i p_y) \end{pmatrix}}, \qquad \psi = \begin{pmatrix} -(p_x - i p_y) \\ p + p_z \\ \frac{cp}{E_+ + mc^2}(p_x - i p_y) \\ -\frac{cp}{E_+ + mc^2}(p + p_z) \end{pmatrix} \times \text{phase}$$

(c) $E = E_- < 0$, $\vec{\Sigma}\cdot\hat{p} = 1$

$$\psi = \begin{pmatrix} \frac{c}{E_- - mc^2}\, p(p + p_z) \\ \frac{c}{E_- - mc^2}\, p(p_x + i p_y) \\ p + p_z \\ p_x + i p_y \end{pmatrix} e^{i(\vec{p}\cdot\vec{x} - E_- t)/\hbar}$$

(d) $E = E_- < 0$, $\vec{\Sigma}\cdot\hat{p} = -1$

$$\cancel{\psi = \begin{pmatrix} -\frac{c}{E_- - mc^2}\, p(p_x - i p_y) \\ \frac{c}{E_- - mc^2}\, p(p - p_z) \\ p_x - i p_y \\ -(p - p_z) \end{pmatrix}} e^{i(\vec{p}\cdot\vec{x} - E_- t)/\hbar}, \qquad \psi = \begin{pmatrix} \frac{cp}{E_- - mc^2}(p_x - i p_y) \\ -\frac{cp}{E_- - mc^2}(p + p_z) \\ -(p_x - i p_y) \\ p + p_z \end{pmatrix}$$

In particular, let $p_x = p_y = 0$, $p = p_z$, then

(a) $E = E_+ = |E|$, $\vec{\Sigma}\cdot\hat{p} = 1$

$$\psi = \begin{pmatrix} 1 \\ 0 \\ cp/(|E| + mc^2) \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x} - |E|t)/\hbar}$$

(b) $E = E_+ = |E|$, $\vec{\Sigma}\cdot\hat{p} = -1$

$$\psi = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -cp/(|E| + mc^2) \end{pmatrix} e^{\text{"}}$$

(c) $E = E_- = -|E|$, $\vec{\Sigma}\cdot\hat{p} = 1$

$$\psi = \begin{pmatrix} -cp/(|E| + mc^2) \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x} + |E|t)/\hbar}$$

(d) $E = E_- = -|E|$, $\vec{\Sigma}\cdot\hat{p} = -1$

$$\psi = \begin{pmatrix} 0 \\ +cp/(|E| + mc^2) \\ 0 \\ 1 \end{pmatrix} \quad \text{"}$$

Discussions

i) positive energy : φ large component

χ small component

negative energy : reverse.

(ii) NR limit. $p \ll mc$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{i(\vec{p}\cdot\vec{x} - |E|t)/\hbar}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad ''$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{i(\vec{p}\cdot\vec{x} + |E|t)/\hbar}$$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad ''$$

If we consider only positive energy, then we have two sol.

$\begin{pmatrix}1\\0\end{pmatrix} e^{i\vec{p}\cdot\vec{x}}$, $\begin{pmatrix}0\\1\end{pmatrix} e^{i\vec{p}\cdot\vec{x}}$

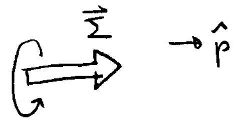
↑ z spin of electron

(iii) $\vec{\Sigma}\cdot\hat{p}$ is called the helicity operator.

$\vec{\Sigma}\cdot\hat{p} = +1$: right-handed state (spin // motion)

$\vec{\Sigma}\cdot\hat{p} = -1$: left-handed state (spin opposite to motion)

RH:



LH

(iv) Properly normalized, the four wave functions are orthonormal:

$$u^{(r)\dagger}(\vec{p})\, u^{(r')}(\vec{p}) = \delta^{rr'}$$

10/1/02 ②

NR limit in a central potential

Consider electrons moving in a central potential $V(r)$, $\vec{A}(r) = 0$ eg. Coulomb potential

Dirac eqn.

$$\left[c\vec{\alpha}\cdot\vec{p} + mc^2\beta + V(r)\right]\psi = E\psi \qquad \vec{p} = -i\hbar\vec{\nabla}$$

To go to NR limit, let

$$E = E' + mc^2 \qquad (\text{NR limit}: E' = E - mc^2 \ll mc^2)$$

and let

$$\psi = \begin{pmatrix}\varphi \\ \chi\end{pmatrix}$$

In Pauli-Dirac representation, we have

$$\begin{cases} c\,\vec{\sigma}\cdot\vec{p}\;\chi = (E' - V)\,\varphi & \text{— ①} \\ c\,\vec{\sigma}\cdot\vec{p}\;\varphi = (2mc^2 + E' - V)\,\chi & \text{— ②} \end{cases}$$

② ⇒ $$\chi = \frac{c\,\vec{\sigma}\cdot\vec{p}}{2mc^2 + E' - V}\,\varphi$$

$$= \frac{1}{2mc}\left(1 + \frac{E'-V}{2mc^2}\right)^{-1}\vec{\sigma}\cdot\vec{p}\;\varphi$$

$$\simeq \frac{1}{2mc}\left(1 - \frac{E'-V}{2mc^2}\right)\vec{\sigma}\cdot\vec{p}\;\varphi$$

substitute into ①, $$\frac{1}{2m}\vec{\sigma}\cdot\vec{p}\left(1 - \frac{E'-V}{2mc^2}\right)\vec{\sigma}\cdot\vec{p}\;\varphi = (E'-V)\,\varphi$$

Simplify, $$\left[\frac{p^2}{2m} - \frac{p^2}{4m^2c^2}E' + \frac{1}{4m^2c^2}(\vec{\sigma}\cdot\vec{p})V(\vec{\sigma}\cdot\vec{p})\right]\varphi = (E'-V)\,\varphi \qquad \text{— ③}$$

Using $$V(\vec{\sigma}\cdot\vec{p}) = (\vec{\sigma}\cdot\vec{p})V + i\hbar\,\vec{\sigma}\cdot\nabla V$$

$$\begin{aligned}(\vec{\sigma}\cdot\vec{p})V(\vec{\sigma}\cdot\vec{p}) &= (\vec{\sigma}\cdot\vec{p})^2 V + i\hbar(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\nabla)V \\ &= p^2 V + i\hbar\left\{\vec{p}\cdot(\nabla V) + i\vec{\sigma}\cdot[\vec{p}\times\nabla V]\right\} \\ &= p^2 V + i\hbar\left\{(\nabla V)\cdot\vec{p} - i\hbar\nabla^2 V + i\vec{\sigma}\cdot\vec{p}\times\frac{\vec{r}}{r}\frac{dV}{dr}\right\} \\ &= p^2 V + \hbar^2\left(\frac{dV}{dr}\frac{\partial}{\partial r} + \nabla^2 V\right) + \hbar\,\vec{\sigma}\cdot\vec{\ell}\,\frac{1}{r}\frac{dV}{dr}\end{aligned}$$

sub into ③, get

$$0 = \left(\frac{p^2}{2m} + V - E'\right)\varphi + \frac{p^2}{4m^2c^2}(V - E')\varphi + \frac{1}{4m^2c^2}\begin{bmatrix} \frac{1}{r}\frac{dV}{dr}\hbar\vec{\sigma}\cdot\vec{\ell} + \hbar^2\nabla^2 V \\ + \hbar\frac{dV}{dr}\frac{\partial}{\partial r}\end{bmatrix}\varphi$$

NR corrections

using $E' - V \approx p^2/2m$ + higher order

$$\left\{\underbrace{\frac{p^2}{2m} + V - \frac{p^4}{8m^3c^2}}_{} + \underbrace{\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{s}\cdot\vec{\ell}}_{} + \frac{\hbar^2}{4m^2c^2}\left(\nabla^2 V + \frac{dV}{dr}\frac{\partial}{\partial r}\right)\right\}\varphi = E'\varphi \qquad - ④$$

Relativistic Correction to K.E.

Spin-orbit Coupling $\xi(r)\,\vec{s}\cdot\vec{\ell}$ (Thomas term)

Remark. last term is not Hermitian!

The problem is because the large component φ is not necessary the wave function Ψ of the NR wave equation.

The important criteria is that $(\Psi,\Psi) = (\varphi,\varphi) + (\chi,\chi)$ must be conserved.

To the order $O(v^2/c^2)$,

$$(\chi,\chi) = \left(\varphi, \left(\frac{\vec{\sigma}\cdot\vec{p}}{2mc}\right)^2\varphi\right) = \left(\varphi, \frac{p^2}{4m^2c^2}\varphi\right)$$

$$(\Psi,\Psi) = \left(\varphi, \left(1 + \frac{p^2}{4m^2c^2}\right)\varphi\right)$$

Therefore the relation should be $\Psi \approx \left(1 + \frac{p^2}{8m^2c^2}\right)\varphi$

or $\varphi \approx \left(1 - \frac{p^2}{8m^2c^2}\right)\Psi$

Ignoring $O(v^4/c^4)$ term, we get from ④

$$\left\{ \frac{p^2}{2m} + V - \frac{p^4}{m^3c^2}\left(\frac{1}{8}+\frac{1}{16}\right) + \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\,\vec{s}\cdot\vec{\ell} + \frac{\hbar^2}{4m^2c^2}\left(\nabla^2 V + \frac{dV}{dr}\frac{\partial}{\partial r}\right)\right\}\Psi$$

$$+(E'-V)\frac{p^2}{8m^2c^2}\Psi = E'\Psi$$

Using $Vp^2 = p^2V + \hbar^2\nabla^2 V + 2\hbar\frac{dV}{dr}\frac{\partial}{\partial r}$

$$\left(\text{pf. } [V,p^2] = [V,\vec{p}]\cdot\vec{p} + \vec{p}\cdot[V,\vec{p}] = i\hbar\nabla V\cdot\vec{p} + i\hbar\vec{p}\cdot\nabla V = \hbar^2\nabla^2 V + 2\hbar\frac{dV}{dr}\frac{\partial}{\partial r}\right.$$

and using $\frac{p^2}{8m^2c^2}(E'-V)\Psi \simeq \frac{p^4}{16m^3c^2}\Psi$

we obtains,

$$\boxed{\left\{\frac{p^2}{2m} + V \underbrace{\underbrace{- \frac{p^4}{8m^3c^2}}_{\substack{\text{Rel. Correction}\\ \text{to KE}\\ V_1}} + \underbrace{\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\,\vec{s}\cdot\vec{\ell}}_{\substack{\text{Thomas term}\\ V_2}} + \underbrace{\frac{\hbar^2}{8m^2c^2}\nabla^2 V}_{\substack{\text{Darwin term}\\ V_3}}}_{\text{Hyperfine structure of energy level.}}\right\}\Psi = E'\Psi \quad + O(v^4/c^4)}$$

- For Hydrogen-like atom, $V = -Ze^2/r$

Thomas term $\xi(r) = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr} = \frac{Ze^2}{2m^2c^2r^3}$

Darwin term $\frac{\hbar^2}{8m^2c^2}\nabla^2 V = \frac{-Z\hbar^2e^2}{8m^2c^2}\nabla^2\frac{1}{r} = \frac{\pi Z\hbar^2e^2}{2m^2c^2}\delta(\vec{r})$

$$\left(\nabla^2\frac{1}{r} = -4\pi\,\delta(\vec{r})\right)$$

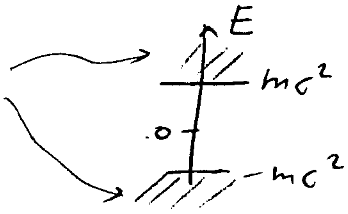
- We have studied the effects of $V_1 + V_2$ using perturbation theory in the last term.

Now we derive them from Dirac eqn, as promised.

Note that the coefficient of Thomas term is half of that as expected from a native derivation (See p. 12 of note for non-relativistic QM)

Hole Theory of Dirac

- Dirac eqn allow soln. of energies
  i.e. Negative energy soln.
  are allowed !



However we know that these energy levels are not really stationary.
In QM, there exists transition between energy levels!
The probability transition rate can be estimated. For the transition from $-mc^2$ to $-2mc^2$,

$$\text{Transition rate} \sim \frac{2\alpha^6}{\pi}\frac{mc^2}{\hbar} \simeq 10^8 \text{ sec}^{-1}$$

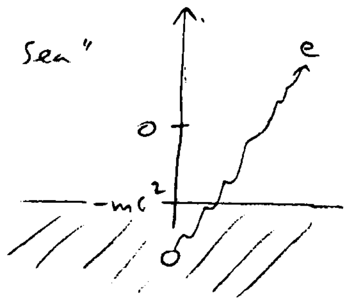
This is too large! The world won't be stable!

- Dirac assumed that:

  All the negative energy states are filled up with electrons and so there cannot be any transition to the negative energy states (due to Pauli exclusion principle).

Consequences :
① Hole theory
② Vaccum polarization

- Hole Theory
  when an electron is excited from the "Dirac Sea"
  A Hole is left behind,
  This is observed as a particle of
  mass $+|E|$
  Charge $+|e|$



Antiparticle of electron , Positron .

Charge conjugation

Thus the wave function $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ $\begin{matrix} \} \text{ positive energy} \longrightarrow e^- \\ \} \text{ negative energy} \longrightarrow e^+ \end{matrix}$

A single wave fun describes both electron & positron. Dirac eqn. not a single particle wave equation.
There is a symmetry between particle + antiparticle.
To see this more explicitly, consider Dirac eqn. in EM field:

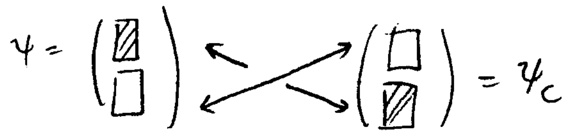
$$(i\hbar \not\partial - e\not A - m)\psi = 0 \qquad — ①$$

If we exchange electron + positron, then the Dirac eqn. should become $(i\hbar\not\partial + e\not A - m)\psi_c = 0$

where ψ_c is the charge conjugate wave function.

We want to find the relation between ψ + ψ_c.

Naively expect:



$$①^*: \left[(i\hbar\partial_\mu + eA_\mu)(\gamma_\mu)^* + m\right]\psi^* = 0$$

If exist matrix M ∍ $\boxed{M\gamma_\mu^* M^{-1} = -\gamma^\mu}$

then $\psi_c = M\psi^*$ will do the job!

To construct M, it is enough to construct M first in a certain representation, eg Pauli-Dirac.

In this reps, $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$, $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore (\gamma^\mu)^* = \gamma^\mu$, $\mu = 0, 1, 3$

$(\gamma^\mu)^* = -\gamma^\mu$, $\mu = 2$

$\therefore M = c_1\gamma^2$ will do the job! $\qquad c_1$ = constant number

$\psi_c = c_1 \gamma^2 \psi^*$

Claim: $\bar{\psi}^T = \gamma^0 \psi^*$

pf. $\bar{\psi} = \psi^+ \gamma^0$ by definition

$\bar{\psi}^T = (\gamma^0)^T \psi^* = \gamma^0 \psi^*$ $(\gamma^0)^T = \gamma^0$ symmetric.

$\therefore \psi_c = c_1 \gamma^2 \gamma^0 \gamma^0 \psi^*$
$= c_1 \gamma^2 \gamma^0 \bar{\psi}^T$

$C = c_1 \gamma^2 \gamma^0$ is usualy called the charge conjugation matrix

c_1 is often taken to be i, then C is a real matrix

$\psi_c = C \bar{\psi}^T = i \gamma^2 \gamma^0 \bar{\psi}^T$

Vaccum polarization

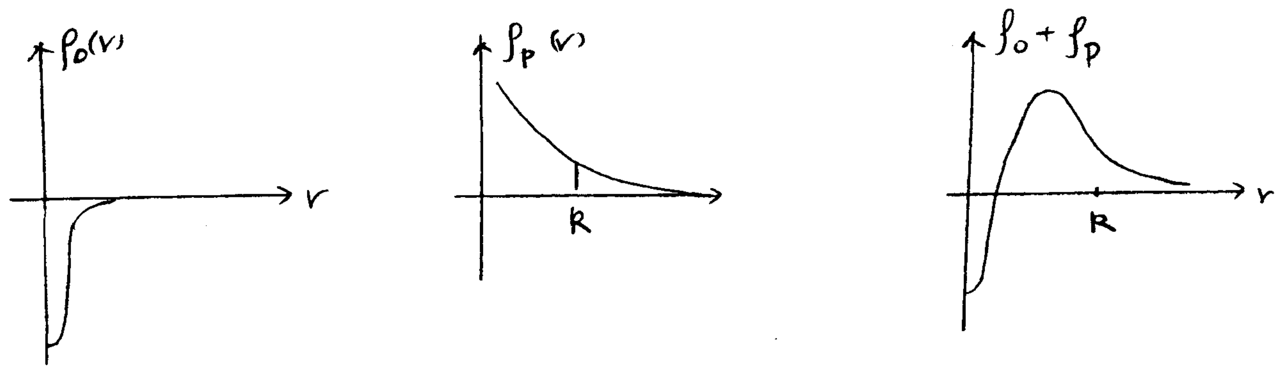
In Dirac theory,

$$\text{vaccum} = \text{sea of electrons filling up all levels } E \leq -mc^2$$

Quantum fluctuations induces electron-positron pairs produced in the vac. Usually the pair couldn't travel long since they recombine and anhillate each other quickly.

However, if we have a electron sit in a vac., the positron may be attracted towards the electron (negatively charged).
As a result, we have effectively a cloud of +ve charge around the electron, → Vaccum polarization

Let $\rho_0(r)$ be the "bare" charge density of electron.
$\rho_p(r)$ be the charge density induced by the e^+e^- cloud from vaccum polarization.



Bare charge $e_0 = \int d^3r\, \rho_0$

observed charge $e = \int d^3r \left[\rho_0 + \rho_p\right]$

$$\boxed{|e_0| > |e|}$$

This is a first example of renormalization.

↳ more details and complete treatment in QFT.