QFT

18/1/02

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- Historically, we have classical mechanics
  then Schrodinger eqn. (Single particle wave eqn.)

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi = i\hbar\frac{\partial}{\partial t}\psi$$

This eqn is nothing but the classical equation

$$\frac{p^2}{2m} + V = E$$

followed by the substitution

$$\vec{p} = -i\hbar\nabla$$
$$E = i\hbar\frac{\partial}{\partial t}$$

Correspondence principle.

However $E = p^2/2m + V$ is a non relativistic energy-momentum relation

people want to generalize QM to the relativistic regime.

This lead to Klein-Gordon eqn., Dirac eqn., etc.

- We dropped KG eqn. since it has problem with negative probability density.

Dirac eqn. has $\rho \geq 0$
relativistic covariant
Good nonrelativistic limit
also predict antiparticle

The last property implies that the wave function of Dirac eqn. no longer describe a single particle. One unavoidably need to introduce antiparticle

Dirac eqn. is not a single particle wave eqn.!

How should one think about ψ ?

At low energy, one can ignore the creation of e^+e^- pair out of the vac. one can therefore focus on the e^- part of the wave fun. (ie 1st two components) This is what we did and in this regime, we arrive at a NR QM with a few correction terms.

At high energy, even if we start with a single electron initially.

ie. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}$ at $t=0$

The creation of e^-e^+ out of vaccum cannot be ignored. The system is a complex mixture of electrons & positrions. The wave function ψ should be able to describe this many bodies system, if Dirac eqn. is correct.

It turn out we need a new interpretation of ψ and the Dirac equation.

we need to promote ψ to become a <u>field</u>, and Dirac eqn. to becomes a <u>field eqn.</u>!

- what does it mean?

  Goal of this next few lectures to explain this.

Very Roughly, $\psi(x)$ is an operator (defined in a "Hilbert space")

$|0\rangle$ represents the Dirac sea (Vaccum).

$\psi(x_1)\,\psi(x_2)\,|0\rangle$ represents a situation with two electrons, one at position x_1, one at x_2.

we imagine that space is divided into a huge # of lattice points labeled by i

$\phi(\vec{x}, t) \rightarrow \phi_i(t)$

	Q.M.	Q.F.T.
Basic coord.	wave function	field. $\phi(x,t)$
Observables	$\phi(x,t)$ wave function	$F[\phi(x,t), t]$ functional of fields
Quantitation	$[x, p] = i\hbar$ (p: momentum conjugate to x.)	$[\phi, \ ] = i\hbar$ (momentum conjugate to field $\phi(x,t)$. It is called $\pi(x,t)$, conjugate field momentum.)

Remarks

1° Because the quantization is performed on $\phi(x,t)$

It is called sometime called second quantization

2° Just as QM is a quantitation of classical mechanics

$$p = \frac{\partial L}{\partial \dot{x}}$$

QFT will be a quantitation of classical field theory

and $\pi(x,t) \doteq \dfrac{\partial L}{\partial \dot{\phi}(x,t)}$

We need to have some basic knowledge of classical field theory first → - Hamiltonian formulation ← more physical.

- Lagrangian formulation ←
  - Good for description of symmetry
  - Good for doing path integral formulation

Lagrangian field theory

- Recall Lagrangian mechanics for a point particle

Lagrangian $L \hat{=} T - V = \frac{1}{2} m \dot{x}^2 - V(x)$

strictly speaking $L[x, \dot{x}]$ is a functional of $x(t)$, $\dot{x}(t)$

trajectory

Action :

$$S \hat{=} \int_{t_1}^{t_2} L(x, \dot{x}) \, dt \qquad \text{fixed } t_1, t_2$$

Action principle : particle move in such a way that S is minimized.

This sounds bizarre But this is how nature works.

Consider a variation in the path,

$$x(t) \to x'(t) = x(t) + a(t) \quad , \quad a \ll x$$

B.C. $a(t_1) = a(t_2) = 0$

$$S \to S' = \int_{t_1}^{t_2} \left[\frac{m}{2}(\dot{x} + \dot{a})^2 - V(x+a)\right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{1}{2} m \dot{x}^2 + m \dot{x} \dot{a} - V(x) - a V'(x)\right] dt + O(a^2)$$

$$= S + \int_{t_1}^{t_2} \left[m \dot{x} \dot{a} - a V'\right] dt$$

$$\hat{=} S + \delta S$$

Since $\displaystyle\int_{t_1}^{t_2} \dot{x} \dot{a} \, dt = \int_{t_1}^{t_2} \left[\frac{d}{dt}(\dot{x} a) - \ddot{x} a\right] dt$

$$= -\int_{t_1}^{t_2} \ddot{x} \, a \, dt + \underbrace{\dot{x} a \Big|_{t_1}^{t_2}}_{0}$$

due to B.C.

$$\therefore \delta S = -\int_{t_1}^{t_2} (m \ddot{x} + V') a \, dt$$

arbitrary fun. of t

$\delta S = 0$ if $m\ddot{x} + V' = 0$ ie Newton's second law
$m\ddot{x} = F = -\frac{dV}{dx}$

eg. SHO, $V = \frac{1}{2} m\omega^2 x^2$

- Lagrangian Field theory (real scalar field)
  $\phi(\vec{x},t)$ real scalar field
  $S = \int \mathcal{L}(\phi, \partial_\mu \phi) \, d^4x$ $\mathcal{L}$ = Lagrangian density

  $L = \int \mathcal{L} \, d^3x$ = Lagrangian
  $S = \int L \, dt$ = Action

Analogous to SHO, the simplest field theory is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2$$

the parameter m will turns out to be the mass of the particles described by the field ϕ after quantization

(Generally, will see:
fields $\xrightarrow{\text{Quantization}}$ particle picture emerges)

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Variational principle and Noether Theorem

consider an action $S = \int \mathcal{L}(\phi, \partial_\mu \phi)\ d^4x$

$\delta S = 0 \Rightarrow$ EOM. $\quad \dfrac{\delta \mathcal{L}}{\delta \phi} - \dfrac{\partial}{\partial x^\mu}\left[\dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right] = 0 \qquad$ (Euler-Lagrange)

pf. $$\delta S = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \underbrace{\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \underbrace{\delta(\partial_\mu \phi)}_{= \partial_\mu(\delta\phi)}}_{= \partial_\mu\left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right] - \partial_\mu\left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right]\delta\phi} \right]$$

$$= \int d^4x \left\{ \frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu\left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right]\right\}\delta\phi$$

$$+ \int d^4x\ \underbrace{\partial_\mu\left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right]}$$

← Boundary term

As in the point particle case, the variation $\delta\phi$ is supposed to vanish at space time infinities.
Since otherwise $\delta\phi$ is arbitrary, we obtain the EOM. //

eg. K-G field. $\quad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$

$$\frac{\delta \mathcal{L}}{\delta \phi} = -m^2\phi$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu\phi$$

$$\therefore\ \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)}\right) - \frac{\delta \mathcal{L}}{\delta\phi} = 0 \Rightarrow \underbrace{\partial_\mu\partial^\mu}_{\Box}\phi + m^2\phi$$

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Noether procedure:

Suppose the action S is continuous under a continuous global symmetry,
(spacetime indep.)

then there exists a conserved current

$J^\mu(x)$

s.t. $\partial_\mu J^\mu = 0$

- First what is a symmetry?

Example: translational invariance

we can shift the origin of coordinate $x^\mu \to x^\mu + a^\mu$ (constant 4-vector)

and there should not be any change in the physical law.

Notice I speak about physical law, not the physical phenomena itself.

In field theory, two kinds of symmetry
- external: spacetime
- internal: acts on the fields themselves, Not on spacetime

- Let $\delta\phi$ be an internal symmetry.

ie, under $\phi \to \phi + \delta\phi$, $\delta S = 0$

ie $\delta \mathcal{L} =$ total derivative $\partial_\mu K^\mu$

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\delta\phi)\\ &= \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\right]}_{\substack{\| \\ 0 \text{ by EOM}}}\delta\phi + \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right]\end{aligned}$$

$$\therefore \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - K^\mu\right) = 0$$

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - K^\mu$$ is a conserved current.

eg. $x^\mu \to x^\mu + a^\mu$ Translation

$\Rightarrow \phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi$

$\mathcal{L}$ is a scalar

$\Rightarrow \mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\mu \partial_\mu (\delta^\mu{}_\nu \mathcal{L})$

$$\therefore J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} a^\nu \partial_\nu \phi - a^\nu \delta^\mu_\nu \mathcal{L}$$

$$= a^\nu T^\mu{}_\nu \qquad \text{where } T^\mu_\nu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

$T^\mu{}_\nu$ is called Energy-momentum tensor

$\partial_\mu T^\mu{}_\nu = 0$

$H = \int T^{00} d^3x$ is a conserved charge $\leftarrow$ associated with time translation

$P^i = \int T^{0i} d^3x$ is a conserved charge $\leftarrow$ associated with spatial translation

$\mathcal{H} = T^{00}$ Hamiltonian density

$\mathcal{P}^i = T^{0i}$ momentum density

$$\frac{dH}{dt} = \int \partial_0 T^{00} d^3x = -\int \partial_i T^{i0} d^3x = 0$$

$$\frac{dp^i}{dt} = \int \partial_0 T^{0i} d^3x = -\int \partial_j T^{ji} d^3x = 0$$

time translation $\Rightarrow$ Energy conservation

Space " $\Rightarrow$ momentum "

eg. K.G field $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\alpha \phi \, \partial^\alpha \phi - m^2 \phi^2)$$

$$\mathcal{H} = T^{00} = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2$$

$$\mathcal{P}^i = T^{0i} = \partial_0 \phi \, \partial_i \phi = \Pi \, \partial_i \phi$$

$\Pi = \partial_0 \phi$ = conjugate field momentum

Complex scalar field

Consider two real scalar fields ϕ_1, ϕ_2 of equal "masses"

free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_1)^2 + \tfrac{1}{2}(\partial_\mu \phi_2)^2 - \tfrac{1}{2}m^2\phi_1^2 - \tfrac{1}{2}m^2\phi_2^2$$

Introducing
$$\begin{cases} \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \end{cases}$$

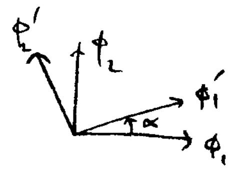
$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - m^2 \phi\phi^*$$

Euler lagrangian EOM: $\dfrac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu\left(\dfrac{\delta \mathcal{L}}{\partial^\mu \phi}\right) = 0 \quad \Rightarrow (\Box + m^2)\phi^* = 0$

(regarding ϕ, ϕ^* as independent) $\dfrac{\delta \mathcal{L}}{\delta \phi^*} - \partial_\mu\left(\dfrac{\delta \mathcal{L}}{\partial^\mu \phi^*}\right) = 0 \quad \Rightarrow (\Box + m^2)\phi = 0$

The Lagrangian is inv. under the transformation:

$$\begin{cases} \phi \to e^{i\alpha}\phi \\ \phi^* \to e^{-i\alpha}\phi^* \end{cases} \Leftrightarrow \begin{cases} \phi_1 \to \phi_1 \cos\alpha + \phi_2 \sin\alpha \\ \phi_2 \to -\phi_1 \sin\alpha + \phi_2 \cos\alpha \end{cases} \qquad \alpha \in \mathbb{R}$$



i.e. it is a rotation in the ϕ_1, ϕ_2 space.

(This is an example of internal symmetry)

To use Noether's construction, we need infinitesimal form of transformation:

$$\begin{cases} \delta\phi = i\alpha\phi \\ \delta\phi^* = -i\alpha\phi^* \end{cases}$$

Noether construction $\Rightarrow J^\mu = \dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}(i\alpha\phi) + \dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)}(-i\alpha\phi^*)$

$$= -\alpha\left[i\phi^*\partial^\mu\phi - i\phi\partial^\mu\phi^*\right]$$

α arbitrary, we have conserved current $j^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$

check conserved? $\partial_\mu j^\mu = i(\phi^* \partial_\mu \partial^\mu \phi - \phi \partial_\mu \partial^\mu \phi^*)$

$= i[\phi^* (-m^2 \phi) - \phi(-m^2 \phi^*)]$ using EoM!

$= 0$

Conserved "charge": $Q = \int j^0 d^3x = i \int \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) d^3x$

remarks

1.° ϕ satisfies the K-G eqn. We tried before to use this eqn. as a relativistic QM eqn. But we rejected it because it has a probability density ρ which is not +ve definite ("j^0 here)

Here we don't think of ϕ as a relativistic wave eqn. We just think of ϕ as a classical field.

There is no particle interpretation yet since we have not quantized the field.

2.° In fact here we will not think of Q as a probability. So we will not reject the K-G field. ie. We keep K-G field theory, but with a new interpretation.

3.° It turns out after quantizing the field,

→ particle picture emerges

→ m^2 becomes $(\text{mass})^2$ of ϕ particle

→ Q becomes electric charge density of ϕ particles.

ie ϕ and ϕ^* will be particle + anti-particle of opposite charge

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A natural question: can one turn the Symmetry to a local one ie $\alpha = \alpha(\vec{x},t)$.

$\phi \to e^{i\alpha}\phi$

$\Rightarrow \phi^*\phi \to \phi^*\phi$ ✓

$\partial_\mu\phi \to e^{i\alpha}(\partial_\mu\phi + i\partial_\mu\alpha\cdot\phi)$

$\therefore |\partial_\mu\phi|^2$ is not inv. ! This is because $\partial_\mu\phi$ does not transform in a simple manner as ϕ.

Therefore, we would like to construct a covariant derivative (like in General rel.,

$D_\mu\phi$ st. $D_\mu\phi \to e^{i\alpha}D_\mu\phi$.

We want $D_\mu\phi$ to be linear in ϕ and contains no more than first order derivative.

$\therefore D_\mu\phi$ takes the form: $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$

Now under: $\phi \to \phi' = e^{i\alpha}\phi$

$A_\mu \to A'_\mu = \cdots$ to be determined

we require $D_\mu\phi \to (D_\mu\phi)' = \partial_\mu\phi' + ieA'_\mu\phi' = e^{i\alpha}D_\mu\phi$

we get: $e^{i\alpha}(\partial_\mu\phi + i\partial_\mu\alpha\,\phi) + ieA'_\mu e^{i\alpha}\phi \stackrel{!}{=} e^{i\alpha}(\partial_\mu\phi + ieA_\mu\phi)$

$\partial_\mu\phi: \quad e^{i\alpha} = e^{i\alpha}$ ✓

$\phi: \quad e^{i\alpha}(i\partial_\mu\alpha + ieA'_\mu) \stackrel{!}{=} e^{i\alpha}\cdot ieA_\mu$

$$\therefore \boxed{A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha}$$

Therefore, the Lagrangian $\mathcal{L} = (D_\mu\phi)(D_\mu\phi)^*$ is inv. under the local transformation

$$\begin{cases} \phi \to \phi' = e^{i\alpha}\phi \\ A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha \end{cases}$$

Also called gauge transformation (Weyl).

Remarks

1.° $D_\mu = \partial_\mu + ieA_\mu$ is called the covariant derivative.
A_μ is the electromagnetic potential

2.° Free scalar field $\longrightarrow$ Coupled to EM field

$$|\partial_\mu \phi|^2 \longrightarrow |D_\mu \phi|^2$$

ie. effectively $\partial_\mu \to \partial_\mu + ieA_\mu$
equivalently $P_\mu = -i\partial_\mu$: $P_\mu \to P_\mu + eA_\mu$ $\leftarrow$ This rule is called "minimal" coupling.

ie. There are nonminimal coupling of EM field to the scalar field. later.

- Now we want to see more explicitly the meaning of $\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2$

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi)^* - ie(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*)A_\mu + e^2 A_\mu A^\mu \phi^*\phi - m^2\phi^*\phi$$

$$= \underbrace{\mathcal{L}_0}_{\text{free Lagrangian}} \underbrace{- e j^\mu A_\mu}_{\mathcal{L}_1} + \underbrace{e^2 A_\mu A^\mu |\phi|^2}_{\mathcal{L}_2} \underbrace{- m^2|\phi|^2}_{\mathcal{L}_3}$$

$\mathcal{L}_1 = -ej^\mu A_\mu$ is a current interacting / coupled to a gauge field.

In term of component, $A_\mu = (\phi, -\vec{A})$
$j^\mu = (\rho, \vec{j})$

$$ej^\mu A_\mu = e(\rho\phi - \vec{j}\cdot\vec{A})$$

Recall this is how a
charge density ρ, electric current $\vec{j}$ } interacts with { electro-static potential ϕ, vector potential $\vec{A}$
in Maxwell theory

Note : for any 4-vector we define the spatial and time component by refering to v^μ. ie
$v^\mu = (v^0, \vec{v})$
$v_\mu = (v^0, -\vec{v})$
$\uparrow$
$\eta^{\mu\nu} = (1, -1, -1, -1)$

$\therefore\ j^\mu$ = EM 4-current.

ie. we identify it as a electric current, not as probability current.

This way we don't have the problem of non-positive-definitness for j^0.

· EoM for $\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2 = (\partial^\mu + ieA^\mu)\phi\,(\partial_\mu - ieA_\mu)\phi^* - m^2|\phi|^2$

Treating ϕ and ϕ^* as independent,

$$\begin{cases} \partial_\mu\left(\dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)}\right) - \dfrac{\delta \mathcal{L}}{\delta \phi^*} = 0 \\[2ex] \partial_\mu\left(\dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right) - \dfrac{\delta \mathcal{L}}{\delta \phi} = 0 \end{cases}$$

complex conjugate of each other

Look at the first one,

$$\partial_\mu(D^\mu\phi) - \left[(D^\mu\phi)\cdot(-ieA_\mu) - m^2\phi\right] = 0$$

$$\text{ie } (\partial_\mu + ieA_\mu)(D^\mu\phi) + m^2\phi = 0$$

$$(D_\mu D^\mu + m^2)\,\phi = 0$$

Q. Is this gauge covariant?

Yes! since $\quad D^\mu\phi \to e^{i\alpha}\cdot D^\mu\phi$

$$\Rightarrow D_\mu(D^\mu\phi) \to e^{i\alpha}\,D_\mu(D^\mu\phi)$$

Also, $\quad m^2\phi \to e^{i\alpha}\,m^2\phi$

$$\therefore (D^\mu D_\mu + m^2)\,\phi = 0 \Rightarrow e^{i\alpha}(D_\mu D^\mu + m^2)\,\phi = 0 \quad ✓$$

- Gauge dynamics

So far A_μ is not dynamical. There is no propagation of A_μ in $\mathcal{L}$
i.e. no kinetic term $(\partial A)^2$.

This can be achieved by adding a kinetic term:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$F_{\mu\nu} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & E_1 & E_2 & E_3 \\ 1 & -E_1 & 0 & -B_3 & B_2 \\ 2 & -E_2 & B_3 & 0 & -B_1 \\ 3 & -E_3 & -B_2 & B_1 & 0 \end{array}$$

$$= -F_{\nu\mu}$$

$$\mathcal{L} = +\frac{1}{2}\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right)$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = \left(-\frac{\partial}{\partial t}\vec{A} - \vec{\nabla}\phi\right)_i = \vec{E}_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\varepsilon_{ijk}\vec{B}_k$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

As a result,

$$\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

EOM of ϕ same as before.

EOM for A_ν: $\partial_\mu\left(\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu}\right) - \frac{\delta \mathcal{L}}{\delta A_\nu} = 0$

$$\partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)}\right) = \partial_\mu\left(-F^{\mu\nu}\right)$$

only from $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu}$

$$\frac{\delta \mathcal{L}}{\delta A_\nu} = -ej^\nu + 2e^2 A^\nu |\phi|^2$$

$$\therefore -\partial_\mu F^{\mu\nu} = -e j^\nu + 2e^2 A^\nu |\phi|^2$$

$$= -ie\left[\phi^* \partial^\nu \phi - \phi \partial^\nu \phi^* + 2ie A^\nu \phi \phi^*\right]$$

$$= -ie\left[\phi^*(\partial^\nu + ieA^\nu)\phi - \phi(\partial^\nu - ieA^\nu)\phi^*\right]$$

$$= -e \mathcal{J}^\nu$$

$$\mathcal{J}^\nu \triangleq i\left[\phi^* D^\nu \phi - \phi (D^\nu \phi)^*\right]$$

$$\boxed{\partial_\mu F^{\mu\nu} = e \mathcal{J}^\nu}$$

Note that $F_{\mu\nu}$ is gauge inv.

pf $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\to \partial_\mu\left(A_\nu - \tfrac{1}{e}\partial_\nu \alpha\right) - \partial_\nu\left(A_\mu - \tfrac{1}{e}\partial_\mu \alpha\right)$$

$$= F_{\mu\nu} - \tfrac{1}{e}\left(\partial_\mu \partial_\nu \alpha - \partial_\nu \partial_\mu \alpha\right)$$

$$= F_{\mu\nu} \quad \checkmark$$

$\mathcal{J}^\nu$ is also gauge inv.

pf. $D^\nu \phi \to e^{i\alpha}(D^\nu \phi)$

$\phi^* \to e^{-i\alpha}\phi^*$ $\checkmark$

Canonical quantization and particle interpretation

Real KG field

- Recall KG eqn has difficulties (i) $\exists$ negative energy sol'n.
  (ii) the current j_μ does not give a +ve definite probability density ρ.
  as a single particle wave eqn.
- What we shall do is to consider KG eqn as describing a field $\phi(x)$.

$\phi(x)$ is a strictly quantum field since the eqn. has no classical analogue.
We will have to quantize the KG field theory $\rightarrow$ first example of QFT.

Aim: 1° Learn how to quantize a field theory
2° How QFT avoid the problems of single particle wave equation.

- Positivity of Energy
  Let us find the energy of the classical KG field. It is obtained from the energy momentum tensor $T^{\mu\nu}$:

$$H = \int d^3x \; T^{00} = \begin{cases} \frac{1}{2}\int (\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2 \, d^3x & \text{real KG field} \\ \int |\partial_0\phi|^2 + |\nabla\phi|^2 + m^2|\phi|^2 \, d^3x & \text{Cpx KG field.} \end{cases}$$

In both cases, the field energy (Hamiltonian) is positive definite!
Thus the scalar field theory is not plagued by the negative energy problem.

But need to ask: How does this field energy related to the energy of single particle states?

Answer : Field Quantization forces us to interpret the field as a Quantum rather than a classical system.
Particle picture emerges.

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Quantization (real KG field)

Step 1: Fourier expansion of ϕ

$$\phi(x) = \int \underbrace{\frac{d^3k}{(2\pi)^3\, 2\omega_k}}\left(a(\vec{k})\, e^{-ikx} + a^+(\vec{k})\, e^{ikx}\right)$$

where $\begin{cases} d^3k = dk_1\, dk_2\, dk_3 \\ k\cdot x = \omega_k t - \vec{k}\cdot\vec{x} \end{cases}$ $\qquad \sqrt{\vec{k}^2 + m^2} \hat{=} \omega_{\vec{k}} > 0$

$a(\vec{k}),\ a^+(\vec{k})$ = Coefficients depending on $\vec{k}$ only.

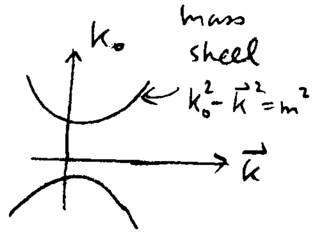
$\to \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}$ = invariant phase space element dV

$$dV = \frac{d^4k}{(2\pi)^4} \cdot \underbrace{2\pi\,\delta(k^2 - m^2)} \cdot \underbrace{\theta(k_0)}$$

imposing the mass shell

conditions:

$\begin{cases} k^2 = k_0^2 - \vec{k}^2 = m^2 \\ k_0 > 0 \end{cases}$



$$\int dV = \int \frac{d^4k}{(2\pi)^3}\, \delta(k_0^2 - \omega_k^2)\, \theta(k_0)$$

$$= \int \frac{d^4k}{(2\pi)^3}\, \delta[(k_0 - \omega_k)(k_0 + \omega_k)]\, \theta(k_0)$$

$$= \int \frac{d^4k}{(2\pi)^3}\, \frac{1}{2k_0}\left[\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)\right]\theta(k_0)$$

$$= \int \frac{d^3k}{(2\pi)^3} \cdot \frac{dk_0}{2k_0}\, \delta(k_0 - \omega_k)\, \theta(k_0)$$

$$= \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2\omega_k}$$

Step 2: Imposing canonical quantization relation

particle : $[x_i, p_j] = i\delta_{ij}$ $\quad i,j = 1,2,3$

$[x_i, x_j] = 0 = [p_i, p_j]$

$\vec{x}, \vec{p}$ refer to the position & momentum measured at the same time.

Analogously, $\phi(\vec{x}, t)$ play the role of $\vec{x}(t)$

and describes a system with an ∞ # of d.o.f., since, at each time, ϕ has an independent value at each point in space.

We can replace the continuum by a lattice of points by dividing the space into cells, each of volume δV_r, and let $\phi_r(t)$ be the average value of $\phi(x)$ in cell r at time t.

Let the average Lagrange density in each cell be $\mathcal{L}_r$. Then the momentum variable p_r, conjugate to ϕ_r, is

$$p_r = \frac{\partial L}{\partial \dot{\phi}_r(t)} = \delta V_r \frac{\partial \mathcal{L}_r}{\partial \dot{\phi}_r(t)} = \delta V_r \cdot \pi_r(t)$$

where we have defined $\pi(\vec{x}, t) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x}, t)}$

and $\pi_r(t)$ is the average value in cell r.

Then the Heisenberg commutation relations give

$$\begin{cases} [\phi_r(t), p_s(t)] = i\delta_{rs} \\ [\phi_r(t), \phi_s(t)] = [p_r(t), p_s(t)] = 0 \end{cases}$$

Since $[\phi_r(t), \pi_s(t)] = \dfrac{1}{\delta V_s} i\delta_{rs}$

In the continuum limit, $\delta V_r \to 0$, we have

$$\begin{cases} [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\,\delta(\vec{x} - \vec{x}') \\ [\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \end{cases}$$

These are known as <u>equal time commutation relations</u> (ETCR).

Step 3 : <u>Particle interpretation</u> .

Define $f_k(x) = \dfrac{1}{((2\pi)^3\, 2\omega_k)^{1/2}}\, e^{-ikx}$

It form an "orthonormal" set :

$$\int d^3x \; f_k^*(x)\, i\overleftrightarrow{\partial_0}\, f_{k'}(x) = \delta^3(\vec{k}-\vec{k}')$$

$$\left|\begin{aligned} & A\overleftrightarrow{\partial_0}B \\ &= A\partial_0 B - (\partial_0 A)B \end{aligned}\right.$$

$$\therefore\ \phi(x) = \int \frac{d^3k}{((2\pi)^3\, 2\omega_k)^{1/2}} \left[f_k(x)\, a(\vec{k}) + f_k^*(x)\, a^+(\vec{k}) \right]$$

It can be inverted to give :

$$\begin{cases} a(\vec{k}) = \int d^3x\; ((2\pi)^3\, 2\omega_k)^{1/2}\; f_k^*(x)\; i\overleftrightarrow{\partial_0}\, \phi(x) \\ \\ a^+(\vec{k}) = \int d^3x\; (\;—\;)^{1/2}\;\; \phi(x)\, i\overleftrightarrow{\partial_0}\, f_k(x) \end{cases}$$

CR of fields $\Rightarrow$ $[a(\vec{k}), a^+(\vec{k}')] = -\int d^3x\, d^3x'\; (2\pi)^3 (4\omega_k\omega_{k'})^{1/2}\,\cdot$

$$\left[f_k^*(x)\, i\partial_0 \phi(x),\; \phi(x')\, i\overleftarrow{\partial_0}\, f_{k'}(x') \right]$$

$$= (2\pi)^3 (4\omega_k\omega_{k'})^{1/2} \int d^3x\, d^3x'\; f_k^*(x)\, \overleftarrow{\partial_0}\, f_{k'}(x')$$

$$[\phi(\vec{x},t),\, \pi(\vec{x}',t)]$$

$$= (2\pi)^3 (4\omega_k\omega_{k'})^{1/2} \int d^3x\;\; f_k^*(x)\, i\overleftrightarrow{\partial_0}\, f_{k'}(x)$$

$$= (2\pi)^3\, 2\omega_k\;\; \delta^3(\vec{k}-\vec{k}')$$

Similarly, we have

$$\boxed{\begin{aligned} & [a(\vec{k}), a(\vec{k}')] = 0 = [a^+(\vec{k}), a^+(\vec{k}')] = 0 \\ & [a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3\, 2\omega_k\, \delta^3(\vec{k}-\vec{k}') \end{aligned}}$$

This is similar to the CR. of SHO! ← crucial to particle interpretation!

In fact, if we expand the KG field as Fourier series

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\, \tilde{\phi}(\vec{p},t)$$

then

$$\Big[\frac{\partial^2}{\partial t^2} + \underbrace{(p^2 + m^2)}_{\omega_p^2}\Big]\phi(\vec{p},t) = 0$$

This is the same as the EOM for a SHO of frequency $\omega_p = \sqrt{p^2+m^2}$

<u>Ex</u> Verify that one can also start with the CR's of a's and derive the ETCR of ϕ & π

- We are now ready to express the Hamiltonian in terms of the oscillators. Starting from H expressed in terms of ϕ & π, we have

$$H = \frac{1}{2}\int d^3x \left[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

and

$$\phi = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p}\left[a(\vec{p})\, e^{-ip^0t + i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})\, e^{ip^0t - i\vec{p}\cdot\vec{x}}\right]$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p}\, e^{i\vec{p}\cdot\vec{x}}\Big[\underbrace{a(\vec{p})\, e^{-i\omega_p t}}_{\text{call it } \alpha(\vec{p})} + \underbrace{a^\dagger(-\vec{p})\, e^{i\omega_p t}}_{\text{call it } \alpha^\dagger(-\vec{p})}\Big] \qquad p^0 = \omega_p$$

fun. of time also.

time dependence understood.

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$$H = \frac{1}{2}\int d^3x \int \frac{d^3\vec{p}\, d^3\vec{p}'}{(2\pi)^6\, 4\omega_p \omega_{p'}}\, e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} \left\{ \begin{array}{l} (-i\omega_p)(-i\omega_{p'})\left[\alpha(\vec{p}) - \alpha^+(-\vec{p})\right]\left[\alpha(\vec{p}') - \alpha^+(-\vec{p}')\right] + \\ \left[(i\vec{p})\cdot(i\vec{p}') + m^2\right]\left[\alpha(\vec{p}) + \alpha^+(-\vec{p})\right]\cdot\left[\alpha(\vec{p}') + \alpha^+(-\vec{p}')\right] \end{array} \right\}$$

$$\int d^3x \to (2\pi)^3 \delta(\vec{p} + \vec{p}')$$

$$= \frac{1}{2}\int d^3\vec{p}\, \frac{1}{(2\pi)^3}\, \frac{1}{4\omega_p^2} \left\{ \begin{array}{l} (\overbrace{-\omega_p^2 + p^2 + m^2}^{0})\left[\alpha(\vec{p})\,\alpha(-\vec{p}) + \alpha^+(-\vec{p})\,\alpha^+(\vec{p})\right] \\ (\underbrace{+\omega_p^2 + p^2 + m^2}_{2\omega_p^2})\left[\alpha(\vec{p})\,\alpha^+(\vec{p}) + \alpha^+(\vec{p})\,\alpha(\vec{p})\right] \end{array} \right\}$$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\cdot\left[\alpha(\vec{p})\,\alpha^+(\vec{p}) + \alpha^+(\vec{p})\,\alpha(\vec{p})\right]$$

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\cdot \omega_p \left\{ a^+(\vec{p})\, a(\vec{p}) + \frac{1}{2}\left[a(\vec{p}), a^+(\vec{p})\right] \right\}}$$

structure of Hamiltonian $= \underbrace{\int \frac{d^3p}{(2\pi)^3\, 2\omega_p}}_{\text{Lorentz invariant measure}} \cdot\ \omega_p \left\{ N(\vec{p}) + \frac{1}{2}\, \delta(0)\, (2\pi)^3 \cdot 2\omega_p \right\}$

∞ Constant

$N(\vec{p})$ = number density operator for the oscillator of momentum $\vec{p}$.
$= a^+(\vec{p})\, a(\vec{p})$

Easy to verify
$$\left[N(\vec{p}), a^+(\vec{p}')\right] = (2\pi)^3\, 2\omega_p\, \delta^3(\vec{p} - \vec{p}')\cdot a^+(\vec{p})$$
$$\left[N(\vec{p}), a(\vec{p}')\right] = -(2\pi)^3\, 2\omega_p\, \delta^3(\vec{p} - \vec{p}')\cdot a(\vec{p})$$

Thus the total number operator

$$N = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\cdot N(\vec{p})$$

satisfies
$$\left[N, a^+(\vec{p})\right] = a^+(\vec{p})$$
$$\left[N, a(\vec{p})\right] = -a(\vec{p})$$

counts the number of oscillator with momentum $\vec{p}$.

Thus we have the picture:

- The field $\phi(\vec{x}, t)$ is decomposed into an infinite collection of oscillators, characterized by momentum $\vec{p}$.

- The occupation number of the oscillator $a(\vec{p})$ is given by the number operator $N(\vec{p})$.

- The energy of the field is given by the total sum of all the oscillators. Each contributes
$$\hbar\omega_p \left(N(\vec{p}) + \frac{1}{2}\delta^{(3)}(0) \right)$$

As in 1 dim SHO, we can build the Fock space of states by using the oscillators: $a(p)$ & $a^\dagger(p)$.

Introduce the vacuum $|0\rangle$ defined by $\begin{cases} a(\vec{p})\,|0\rangle = 0 & \forall \vec{p}. \\ \langle 0|0\rangle = 1 \end{cases}$

Then define <u>Fock Space</u> as the space of all states of the form
$$a^\dagger(k_1)\cdots a^\dagger(k_n)\;|0\rangle \qquad n = 1, 2, \cdots$$

The state $a^\dagger(k)\,|0\rangle$ is a state containing one particle of energy ω_k.

$$N\, a^\dagger(k)\,|0\rangle = a^\dagger(k)\,|0\rangle \qquad H\, a^\dagger(k)\,|0\rangle = (\omega_k + c)\, a^\dagger(k)\,|0\rangle$$

The state $\underbrace{a^\dagger(k_1)\cdots a^\dagger(k_n)|0\rangle}_{|\psi\rangle}$ is a state containing n particles with total energy $\sum_{i=1}^{n}\omega_{k_i}$

$$N|\psi\rangle = n\,|\psi\rangle \qquad H|\psi\rangle = \left(\sum_{i=1}^{n}\omega_{k_i} + c'\right)|\psi\rangle$$

$c = c' =$ same infinite constant!

$$c = \frac{1}{2}\int d^3p\;\omega_p\;\delta^{(3)}(0)$$

Note: 1. the state $a^+(k)|0\rangle$ are not normalizable
$$\langle 0|a(k)\, a^+(k)|0\rangle = (2\pi)^3\ 2\omega_k\, \delta^3(0)$$

This is not surprising!
Because $a^+(k)$ create a particle of definite energy and momentum.
Heisenberg uncertainity principle says that such a particle has no definite position. The wave function is a plane wave
We know from QM that such states are nonnormalizable!

To obtain a normalizable state, we build a wave packet by superposition
The state $|\psi\rangle = \int d^3\vec{k}\, f(\vec{k})\, a^+(\vec{k})\, |0\rangle$
will be normalizable if $\int d^3k\, |f(k)|^2 \triangleq N_\psi = \text{norm} < \infty$.
Also, prob. distribution $\rho = \langle\psi|\psi\rangle / N_\psi$ is positive definite!

2. The above remark applies to all the states in the Fock space.

3. The bare vac. $|0\rangle$ is an eigenstate of the Hamiltonian, of energy

$$H|0\rangle = \underbrace{\frac{1}{2}\int d^3p\ \omega_p\ \delta^3(0)}_{\text{Constant } c \text{ we refered to before.}}\ |0\rangle \triangleq E_0\,|0\rangle$$

E_0 is called the zero point energy.
– Note that $\int d^3x = (2\pi)^3\,\delta^{(3)}(0)$ $\quad \left(\because\ \delta^{(3)}(k) = \frac{1}{(2\pi)^3}\int d^3x\ e^{ikx}\right)$

$$E_0 = \frac{V\cdot\int d^3p}{(2\pi)^3}\cdot\frac{1}{2}\omega_p \underset{\text{restoring } \hbar}{=} \frac{V\cdot\int d^3p}{(2\pi\hbar)^3}\cdot\frac{1}{2}\hbar\omega_p$$

$V\cdot\int d^3p$ = phase space vol. of the system

$(2\pi\hbar)^3$ = minimal unit of phase space volume
$$\Delta x\,\Delta y\,\Delta z\ \Delta p_x\,\Delta p_y\,\Delta p_z \geq (2\pi\hbar)^3$$

The ratio $\int \frac{d^3x \, d^3p}{(2\pi\hbar)^3}$ counts the number of quantum cells one can divide the total phase space volume of the system.

Each such cell is associated an SHO of frequency ω_p.

$\therefore$ E_0 has the physical meaning of:

E_0 = Sum of all zero point energy of the oscillators of the theory.

- E_0 is a constant

  in the sense it is independent of the state

  ie. $H|\psi\rangle = (E_\psi + E_0)|\psi\rangle$

  E_ψ: part depending on $|\psi\rangle$; E_0: indep. of $|\psi\rangle$

  for any eigenstate $|\psi\rangle$ of H.

- $E_0 = \infty$

- Is E_0 measurable?

  No. We can get rid of E_0 by redefining the origin of the energy scale.
  Note that absolute energy has no meaning in physics.
  Only relative change of energy has a measurable meaning.

  Thus we will get rid of E_0 by redefining the zero point of the energy scale.

  E_0 is not measurable, not even for gravity!

- We will use the Hamiltonian $\hat{H} = H - \langle 0|H|0\rangle$ by subtracting out the infinite constant.
  This procedure is called <u>Normal ordering</u>.

Normal ordering of operators

Normal ordering means all creation operators are to appear to the left of destruction operators.

It is denoted by placing colons on both sides of the operator product.

e.g. $: a(k_1)\, a^{+}(k_2) : \; = \; : a^{+}(k_2)\, a(k_1) : \; = \; a^{+}(k_2)\, a(k_1)$

Note that when $k_1 \neq k_2$, $a(k_1)$ & $a^{+}(k_2)$ commute

when $k_1 = k_2$, $a(k)\, a^{+}(k) = a^{+}(k)\, a(k) + (2\pi)^3\, 2\omega_k\, \delta^{(3)}(0)$

Normal ordering means dropping this constant!

$$\hat{H} = :H: = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[a^{+}(p)\, a(p)\, \omega_p \right]$$

$\langle 0 | :H: | 0 \rangle = 0$

Note: E_0 is the first infinity we encountered in QFT. It is not measurable!

We get rid of this by a simple normal ordering.

$\exists$ other infinities in QFT which cannot be removed this way.

But are removed by a procedure called renormalization

The renormalized theory can still sense the presence of ∞.

So these ∞ are measurable.

19/2/02 (10)

Recap: ① We started with a classical field theory of KG Scalar field.

② We did a Fourier decomposition of φ in terms of a & a^+

relativistic invariant since we used the relativistic inv. measure

$$d\mu \triangleq \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}.$$

③ By discretizing the Spatial volume, we imposed the usual quantum mechanical CR. $[q_i, p_j] = i\delta_{ij}$

④ The Continuum limit yields the ETCR of $\varphi(\vec{x},t)$ and $\pi(\vec{x},t)$

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\varphi(\vec{x},t), \varphi(\vec{y},t)] = 0$$

$$[\pi(\vec{x},t), \pi(\vec{y},t)] = 0$$

⑤ These ETCR implies CR for the operators a, a^+, implying that they are SHO.

⑥ Using the a, a^+, one then build the Fock space $\mathcal{F}$, whose states have clear multi particles interpretation. These particle states have <u>+ve definite probability distribution</u> ρ!

⑦ The energy of these particle states are measured by the normal ordered Hamiltonian $:H:$

<u>Energy is positive definite</u>!

Thus, the two problems that has troubled the <u>KG wave QM</u> are solved by treating KG eqn as a <u>QFT</u>.

<u>Final Remark</u>: We started out with an action S which is Lorentz inv.

To quantize the system, we impose ETCR. Time is singled out.

NEED to check the QFT thus obtained respect Lorentz symmetry.

How? → See Exercise

Free Spinor Field Theory

The Dirac eqn. is $(i\not\partial - m)\psi = 0$

can be obtained from $\mathcal{L}$

$$\mathcal{L} = \frac{i}{2}\,\bar\psi\,\gamma^\mu \overleftrightarrow{\partial_\mu}\psi - m\bar\psi\psi$$
$$= \frac{i}{2}\left(\bar\psi\,\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\,\gamma^\mu\psi\right) - m\bar\psi\psi \qquad \leftarrow \text{Hermitian}$$

$$0 = \frac{\delta\mathcal{L}}{\delta\bar\psi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar\psi)} = \frac{i}{2}\gamma^\mu\partial_\mu\psi - m\psi + \frac{i}{2}\partial_\mu(\gamma^\mu\psi) = (i\gamma^\mu\partial_\mu - m)\psi$$

$$S = \int d^4x\ \mathcal{L} = \int d^4x\ \frac{i}{2}(\bar\psi\gamma^\mu\partial_\mu\psi - \partial_\mu\bar\psi\,\gamma^\mu\psi) - m\bar\psi\psi$$
$$= \int d^4x\ \underbrace{\bar\psi(i\gamma^\mu\partial_\mu - m)\psi}_{\text{more often seen in this form.}} \qquad \text{By parts}$$

$$\pi = \frac{\delta\mathcal{L}}{\delta\dot\psi} = i\psi^\dagger$$

$$H = \int d^3x(\pi\dot\psi - \mathcal{L}) = \int d^3x\ i\psi^\dagger\dot\psi - \psi^\dagger\gamma^0(i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi)$$
$$= \int d^3x\ \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi$$
$$= \int d^3x\ \psi^\dagger(-i\vec\alpha\cdot\nabla + \beta m)\psi \qquad \vec\alpha = \gamma^0\vec\gamma,\quad \beta = \gamma^0$$
$$= \int d^3x\ \psi^\dagger\gamma^0(i\gamma^0\partial_0\psi)$$
$$= \int d^3x\ i\psi^\dagger\frac{\partial}{\partial t}\psi$$

NB. Unlike KG, H is not positive definite.

In KG, H is positive definite

Here, H is not positive definite yet.

Negative energy difficulty is not removed by treating Dirac eqn. as a field eqn., as it was in KG. It is only removed on quantization.

11/12/02 ⑪

Quantization

As in KG, we expand Dirac eqn. in terms of the plane wave soln as follows:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \sum_{\alpha=1,2} \left[b_\alpha(k)\, u^{(\alpha)}(k)\, e^{-ikx} + d_\alpha^+(k)\, v^{(\alpha)}(k)\, e^{ikx} \right]$$

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \sum_{\alpha=1,2} \left[b_\alpha^+(k)\, \bar{u}^{(\alpha)}(k)\, e^{ikx} + d_\alpha(k)\, \bar{v}^{(\alpha)}(k)\, e^{-ikx} \right]$$

where the plane wave spinors:

at rest

$$u^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\psi(x) = u(0)\, e^{-imt}$ positive energy soln.

$\psi(x) = v(0)\, e^{imt}$ negative energy soln.

momentum p

$$u^{(1)}(p) = \eta \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix}, \quad u^{(2)}(p) = \eta \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \qquad \text{soln. of } (\not{p} - m)\, u(p) = 0$$

$$v^{(1)}(p) = \eta \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)}(p) = \eta \begin{pmatrix} \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad \text{soln. of } (\not{p} + m)\, v(p) = 0$$

$$p_\pm \triangleq p_x \pm i p_y \qquad E \triangleq \sqrt{p^2 + m^2}$$

The normalization η are choosen s.t. $\bar{u}^i u^j = \delta_{ij} = -\bar{v}^i v^j$

This implies $\to |\eta|^2 \cdot \left\{ 1 - \frac{p^2}{(E+m)^2} \right\} = |\eta|^2 \left(\frac{2m}{E+m} \right) \Rightarrow |\eta| = \sqrt{\frac{E+m}{2m}}$

$\to u^{i+} u^j = v^{i+} v^j = \frac{E}{m} \delta_{ij}$

NB. 1° These wave function $u^{(\alpha)}$, $v^{(\alpha)}$ are related to those on p. 27 of the note on Rel. QM. by linear superposition.

e.g. on p. 27 there,

$$\psi^{(a)} - \chi^{(b)} = \begin{pmatrix} 2p \\ 0 \\ 2p\, p_z/E+m \\ 2p\, p_+/E+m \end{pmatrix} = 2p \underbrace{\begin{pmatrix} 1 \\ 0 \\ p_z/E+m \\ p_+/E+m \end{pmatrix}}$$

← our $u^{(1)}$ here.

We prefer to use the basis $\{u^{(\alpha)}, v^{(\alpha)}\}$ as they correspond immediately to a clear particle picture when the particle is at rest.

2° Note that we have 2 sets of coefficients : $b_\alpha(k)$, $d_\alpha(k)$

On quantization, these will become two set of oscillators ("fermionic") :

$\begin{cases} \text{One create and annihilate electrons } e^- \\ \text{——————————— positrons } e^+ . \end{cases}$

3° The Spinor wave functions satisfy :

Normalization: $\begin{cases} \bar{u}^{(\alpha)}(p)\, u^{(\alpha')}(p) = \delta_{\alpha\alpha'} \\ \bar{v}^{\alpha}(p)\, v^{(\alpha')}(p) = -\delta_{\alpha\alpha'} \\ \bar{u}^{(\alpha)}(p)\, v^{(\alpha')}(p) = 0 \end{cases}$ $\Leftrightarrow$ $u^{(\alpha)\dagger}(p)\, u^{(\alpha')}(p) = v^{(\alpha)\dagger}(p)\, v^{(\alpha')}(p) = \frac{E}{m}\delta_{\alpha\alpha'}$

EOM : $\begin{cases} (\not{p} - m)\, u(p) = 0 \\ (\not{p} + m)\, v(p) = 0 \end{cases}$ $\Leftrightarrow$ $\begin{cases} \bar{u}(p)(\not{p} - m) = 0 \\ \bar{v}(p)(\not{p} + m) = 0 \end{cases}$

completeness relation: $\begin{cases} \sum_\alpha u_i^{(\alpha)}(p)\, \bar{u}_j^{(\alpha)}(p) = \left(\frac{\not{p}+m}{2m}\right)_{ij} \\ \sum_\alpha v_i^{(\alpha)}(p)\, \bar{v}_j^{(\alpha)}(p) = \left(\frac{\not{p}-m}{2m}\right)_{ij} \end{cases}$ ← orthonormal projectors

- Q: What should be the commutation relations for b's & d's ?

We substitute ψ, $\bar\psi$ into the Hamiltonian

$$H = \int d^3x \; \psi^+ i \frac{\partial}{\partial t} \psi \quad = \int d^3x \; \frac{i}{2}\left[\bar\psi \frac{\partial}{\partial t}\psi - \frac{\partial}{\partial t}\bar\psi \; \psi\right]$$

$$= \int d^3x \sum_{\alpha,\alpha'} \frac{i}{2} \int \frac{d^3k \, d^3k'}{(2\pi)^6} \frac{m^2}{k_0 k_0'} \cdot$$

$$\left\{ \begin{aligned} &\left[b^+_\alpha(k)\, u^{+(\alpha)}(k)\, e^{ikx} + d_\alpha(k)\, v^{+(\alpha)}(k)\, e^{-ikx}\right]\left[b_{\alpha'}(k')\, u^{(\alpha')}(k')(-ik_0')\, e^{-ik'x} + d^+_{\alpha'}(k')\, v^{(\alpha')}(k')(ik_0')\, e^{ik'x}\right] \\ &- \left[b^+_\alpha(k)\, u^{+(\alpha)}(k)(ik_0)\, e^{ikx} + d_\alpha(k)\, v^{+(\alpha)}(k)(-ik_0)\, e^{-ikx}\right]\left[b_{\alpha'}(k')\, u^{(\alpha')}(k')\, e^{-ik'x} + d^+_{\alpha'}(k')\, v^{(\alpha')}(k')\, e^{ik'x}\right] \end{aligned} \right\}$$

Using $\int d^3x \, e^{i(k-k')x} = (2\pi)^3 \underbrace{\delta^{(3)}(k-k')}_{\hookrightarrow \text{ enforce } k_0 = k_0'} e^{i(k_0 - k_0')t} = (2\pi)^3 \delta^{(3)}(k-k')$

$b^+ b$, $d d^+$ terms add up.
$b^+ d^+$, $d b$ terms cancel out

We get $H = \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \cdot k_0 \sum_\alpha \left[b^+_\alpha(k)\, b_\alpha(k) - d_\alpha(k)\, d^+_\alpha(k)\right]$

If we assume commutation relation for b & d as (roughly):

$$[b, b^+] = 1$$
$$[d, d^+] = 1$$

then H is <u>not</u> positive definite ! The d-quanta will contribute a negative amount.

The only way to avoid this is to introduce the notion of anticommutator:

$$\{A, B\} = AB + BA,$$

and to postulate the anticommutation relations,

$$\begin{cases} \{ b_\alpha(k),\ b^\dagger_{\alpha'}(k') \} = \{ d_\alpha(k),\ d^\dagger_{\alpha'}(k') \} = (2\pi)^3 \frac{k_0}{m} \cdot \delta^{(3)}(k-k')\ \delta_{\alpha\alpha'} \\ \{ b_\alpha(k),\ b_{\alpha'}(k') \} = \{ d_\alpha(k),\ d_{\alpha'}(k') \} = 0 \end{cases}$$

First proposed by Jordan & Wigner.

- Normal ordered Hamiltonian

$$H = \int d^3x\ :\psi^\dagger i \frac{\partial}{\partial t} \psi:$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \cdot k_0 \sum_\alpha \left[b^\dagger_\alpha(k)\, b_\alpha(k) + d^\dagger_\alpha(k)\, d_\alpha(k) \right]$$

is now positive definite.

- Fock space:

Define the vac $|0\rangle$ by $b_\alpha(p)\,|0\rangle = d_\alpha(p) = 0$

The Fock space is obtained by acting on the vac. with any number of $b^\dagger_\alpha(k)$ & $d^\dagger_\alpha(k)$

ie $|\psi\rangle = b^\dagger_{\alpha_1}(k_1) \cdots b^\dagger_{\alpha_n}(k_n)\ d^\dagger_{\beta_1}(q_1) \cdots d^\dagger_{\beta_m}(q_m)\,|0\rangle$

Note: 1° due to the anti-commutation relations, $|\psi\rangle$ is antisymm. w.r.t. exchange of any pair of $(\alpha_i, k_i) \leftrightarrow (\alpha_j, k_j)$
or $(\beta_i, q_i) \leftrightarrow (\beta_j, q_j)$

This is the Fermionic statistics! Anti-Commutation relation $\Rightarrow$ Pauli Exclusion principle.
This is how symmetrization postulate of QM is built into QFT!

2° The state $|\psi\rangle$ has the interpretation of having created from the vac. $|0\rangle$: n electrons of mom. $k_1, \cdots, k_n$
and m positrons ------ $q_1, \cdots, q_m$

Charge:

$$Q = \int d^3x \; : j_0(x) :$$

$$= \int d^3x \; : \psi^+(x)\, \psi(x) :$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \cdot \sum_\alpha \left[b_\alpha^+(k)\, b_\alpha(k) - d_\alpha^+(k)\, d_\alpha(k) \right]$$

Thus, b^+ create "particles" of positive charge.
d^+ ---- "antiparticles" of negative charge.

Equal-time anti-commutation relations for fields

$$\{\psi_i(x,t), \psi_j^+(x',t)\} = \sum_{\alpha\alpha'} \int \frac{d^3k\, d^3k'}{(2\pi)^6} \cdot \frac{m^2}{k_0 k_0'} \left\{ \begin{aligned} & u_i^{(\alpha)}(k)\, \bar{u}_\ell^{(\alpha')}(k')(\gamma^0)_{\ell j} \underbrace{\{b_\alpha(k), b_{\alpha'}^+(k')\}}\, e^{-ikx+ik'x'} \\ & + v_i^{(\alpha)}(k)\, \bar{v}_\ell^{(\alpha')}(k')\,(\gamma^0)_{\ell j} \underbrace{\{d_\alpha^+(k), d_{\alpha'}(k')\}}_{(2\pi)^3 \frac{k_0}{m} \delta^{(3)}(k-k')\,\delta_{\alpha\alpha'}}\, e^{ikx - ik'x'} \end{aligned} \right\}$$

$i, j = 1, \cdots, 4$

$$= \sum_\alpha \int \frac{d^3k}{(2\pi)^3} \frac{m}{k_0} \left\{ u_i^{(\alpha)}(k)\, \bar{u}_\ell^{(\alpha)}(k)\, \gamma^0_{\ell j}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + v_i^{(\alpha)}(k)\, \bar{v}_\ell^{(\alpha)} \gamma^0_{\ell j}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k_0} \left[(\not{k}+m)\gamma^0\right]_{ij} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + \left[(\not{k}-m)\gamma^0\right]_{ij} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

changing $\vec{k} \to -\vec{k}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} \cdot 2k_0 \delta_{ij} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= \delta^{(3)}(\vec{x}-\vec{x}')\, \delta_{ij}$$ ← Note: no need of "i"

Similarly, $\{\psi_i(x,t), \psi_j(y,t)\} = 0$

$$\{\psi_i^+(x,t), \psi_j^+(y,t)\} = 0$$

<u>Time ordering and propagators</u>

<u>Complex KG Scalar</u>

- Consider the quantum mechanical propagation of a particle at x to x'.
  The state describing the particle located at $x=(\vec{x},t)$ is $|\psi(\vec{x},t)\rangle$
  ———————————————— $x'=(\vec{x}',t')$ is $|\psi(\vec{x}',t')\rangle$

  The amplitude for the propagation is the overlap between the two states
  $\langle \psi(\vec{x}',t') \,|\, \psi(\vec{x},t)\rangle$
  Square of this amplitude is the probability that this process will occur.

- Now consider the propagation of a charged scalar field theory.
  The state corresponding to a particle of charge $+1$ at x is $\varphi^*(x)|0\rangle$.
  ———————————————————————— x' is $\varphi^*(x')|0\rangle$.
  Thus the quantum mechanical amplitude to transport the charge from x to x' is
  $$\langle 0|\varphi(x')\varphi^*(x)|0\rangle$$

  We can interpret this as the propagation of a $+1$ charge, created out of vacuum at x, transport from x to x', and then reabsorbed into vacuum at x'.
  Since we can't absorb the particle before it is created, this process only makes sense if $t'>t$.

  This is not the TOTAL amplitude for the propagation of the $+1$ charge.
  The total amplitude is the sum of all the amplitudes of different processes that give equivalent physical results.

  In the process above, the charge at x is increased by one unit.
  ——————— x' is decreased by one unit.
  We can achieve the same thing by creating a particle of charge -1 at x', transport it to x, then destroy it.
  The corresponding amplitude is

  $$\langle 0|\varphi^*(x)\,\varphi(x')|0\rangle$$ make sense only if $t>t'$.

Thus the total amplitude $G(x',x)$ for a transport of charge +1 from x to x' is

$$G(x',x) = \theta(t'-t)\langle 0|\varphi(x')\varphi^*(x)|0\rangle + \theta(t-t')\langle 0|\varphi^*(x)\varphi(x')|0\rangle$$

We can write this in a more compact form by introducing the Dyson time-ordering operator T:

$$T\, A(x)\, B(x') = \begin{cases} A(x)\; B(x') & \text{if } t > t' \\ B(x')\; A(x) & \text{if } t' > t \end{cases}$$

then

$$G(x',x) = \langle 0|T\,\varphi(x')\,\varphi^*(x)\;|0\rangle$$ This is called <u>Propagator</u>.

Using
$$\begin{cases} \varphi(x) = \displaystyle\int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\cdot\left(a(k)\, e^{-ikx} + b^+(k)\, e^{ikx}\right) \\ \varphi^*(x) = \displaystyle\int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left(b(k)\, e^{-ikx} + a^+(k)\, e^{ikx}\right) \end{cases}$$

$$[a(k), a^+(k')] = [b(k), b^+(k')] = (2\pi)^3\, 2\omega_k\, \delta^{(3)}(k-k')$$
all others vanishing.

one get $$G(x',x) = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[\theta(t'-t)\, e^{-ik\cdot(x'-x)} + \theta(t-t')\, e^{ik\cdot(x'-x)}\right]$$

Check it!

We can write it in a covariant form if we use the integral reps. of the Heaviside function

$$\theta(t) = \lim_{\epsilon\to 0^+}\int \frac{d\omega}{2\pi i}\;\frac{e^{i\omega t}}{\omega - i\epsilon}$$

$i\epsilon$: if $t>0$, we close the contour above the $Re(\omega)$ axis enclosing the pole.

if $t<0$, we close the contour in the bottom half plane, missing the pole. So the integral vanishes.

Thus, $G(x',x) = \lim_{\epsilon\to 0^+} -i \int \frac{d^3k}{(2\pi)^4 2\omega_k} \int \frac{d\omega}{\omega - i\varepsilon} \left(e^{i(\omega-\omega_k)(t'-t)} e^{i\vec{k}\cdot(\vec{x}'-\vec{x})} + c.c. \right)$

Now change variable in first integral from ω to $k_0 = \omega_k - \omega$.
2nd integral from ω to $k_0 = \omega - \omega_k$ & $\vec{k}$ to $-\vec{k}$.

then $$G(x',x) = \lim_{\epsilon\to 0^+} -i \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\omega_k} \cdot e^{-ik\cdot(x'-x)} \underbrace{\left(\frac{1}{\omega_k - k_0 - i\varepsilon} + \frac{1}{k_0 + \omega_k - i\varepsilon} \right)}_{\frac{2\omega_k - 2i\varepsilon}{(\omega_k - i\varepsilon)^2 - k_0^2}}$$

$$\simeq \frac{-2\omega_k}{k^2 - m^2 + i\varepsilon} + O(\varepsilon^2)$$

became positive

$$= \lim_{\epsilon\to 0^+} \cdot i \int \frac{d^4k}{(2\pi)^4} \cdot \frac{e^{-ik\cdot(x'-x)}}{k^2 - m^2 + i\varepsilon}$$

$$\equiv i\Delta_F(x'-x)$$

Δ_F satisfies. $(\Box' + m^2)\Delta_F(x'-x) = -\delta^{(4)}(x'-x)$

Thus $G(x',x)$ is a Green function for the KG eqn.!

$\boxed{\text{Propagator in QFT} = \text{Green function}}$

Vacuum expectation value (VEV)
of time ordered product of Quantum fields

NB. For real scalar field ,
$\langle 0| T\,\phi(x)\phi(y)\,|0\rangle = i\Delta_F(x-y)$ Same Δ_F !

<u>Dirac fermion</u>

For fermions, due to their anticommuting nature, we introduce the time ordering

$$T\, A(x)\, B(x') = \begin{cases} A(x)\, B(x') & t > t' \\ -B(x')\, A(x) & t' > t \end{cases} \qquad A, B = \psi \text{ or } \psi^{+}$$

The propagator is defined as

$$i S_F(x', x) = \langle 0 | T\, \psi(x')\, \bar{\psi}(x)\, | 0 \rangle$$

S_F is a matrix. In matrix notation

$$i S_F(x', x)_{ij} = \langle 0 | T\, \psi_i(x')\, \bar{\psi}_j(x)\, | 0 \rangle$$

Using the expansion of $\psi, \bar{\psi}$; anticommutation relation of $b(k)$, $d(k)$ and

$$\sum_\alpha u_i^{(\alpha)}(p)\, \bar{u}_j^{(\alpha)}(p) = \left(\frac{\not{p} + m}{2m}\right)_{ij}, \qquad \sum_\alpha v_i^{(\alpha)}(p)\, v_j^{(\alpha)}(p) = \left(\frac{\not{p} - m}{2m}\right)_{ij}$$

one can obtain

$$i S_F(x', x) = \int \frac{d^4 p}{(2\pi)^4}\, e^{-ip\cdot(x'-x)}\, \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

$$= (i\not{\partial} + m)\, \Delta_F(x' - x)$$

scalar propagator.

S_F satisfies:

$$(i\not{\partial} - m) S_F(x) = \delta^{(4)}(x)$$

Fourier transform of propagators:

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon}$$

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} = \frac{1}{\not{p} - m + i\varepsilon}$$

5/3/02

(14)

S matrix expansion in Interaction Representation

Interaction reps.

In Schrodinger reps. state vector $\Phi^{(S)}(t)$ evolves by

$$i\frac{\partial}{\partial t}\Phi^{(S)} = H^{(S)}\Phi^{(S)}$$

$H^{(S)}$ = Hamiltonian in Schrodinger reps.

$$= \underbrace{H_0^{(S)}}_{\text{free Hamiltonian}} + \underbrace{H_I^{(S)}}_{\text{perturbation / interaction}}$$

Now change basis to $\Phi^{(I)}(t) = e^{iH_0^{(S)}t}\,\Phi^{(S)}$

then $O^{(I)}(t) = e^{iH_0^{(S)}t}\,O^{(S)}\,e^{-iH_0^{(S)}t}$

(I) stands for "interaction reps."

In this reps, $i\dfrac{\partial\Phi^{(I)}}{\partial t} = i\left[iH_0^{(S)}e^{iH_0^{(S)}t}\Phi^{(S)} + e^{iH_0^{(S)}t}\dfrac{\partial\Phi^{(S)}}{\partial t}\right]$

$$= \cancel{-H_0^{(S)}e^{iH_0^{(S)}t}\Phi^{(S)}} + e^{iH_0^{(S)}t}\underbrace{H^{(S)}}_{\cancel{H_0^{(S)}}+H_I^{(S)}}e^{-iH_0^{(S)}t}\Phi^{(I)}$$

$$= \underbrace{e^{iH_0^{(S)}t}H_I^{(S)}e^{-iH_0^{(S)}t}}_{H_I^{(I)}}\cdot\Phi^{(I)}$$

As for operator $O^{(I)}(t)$,

$$\frac{d}{dt}O^{(I)}(t) = i\left[H_0^{(S)}, O^{(I)}\right] \Leftarrow$$

$\because O^{I}(0) = O^{(S)}$

$\therefore O^{(I)}(t) = e^{iH_0^{(S)}t}O^{I}(0)e^{-iH_0^{(S)}t}$

Dropping the superscript (I) for interaction picture, we have

$$\begin{cases} i\dfrac{\partial}{\partial t}\Phi = H_I\Phi & \text{— ①} \\ \dfrac{d}{dt}O = i[H_0, O] & \text{— ②} \end{cases}$$

ie. state vector evolves according to H_I

Operator " " " H_0

· Interaction picture is particularly convenient for QFT since field operator (φ, ψ etc) satisfies <u>free field eqn.</u> Even in the presence of interaction!!!

Thus (i), quantization procedure we carried out before will go through in the interaction picture!

(ii). We still have a particle picture described by Fock space!

However, a state $\Phi^{(I)}$ will evolve according to H_I. (eqn ①)

Don't be confused.
Not an operator

To understand how particle states evolve under the influence of H_I we need to know how Φ transform.

This will tell us how particles scatter from each other!

(of course, when $H_I = 0$, there is no scattering since particles are free

Fortunately, we can solve ① !

<u>U matrix and S matrix</u>

Define an <u>operator</u> $U(t, t_0)$ by

$$\Phi(t) = U(t, t_0)\, \Phi(t_0) \qquad\qquad U(t_0, t_0) = 1$$

then ① reduces to

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I(t)\, U(t, t_0)$$

$$\Rightarrow \quad U(t, t_0) = 1 - i \int_{t_0}^{t} dt\, H_I(t)\, U(t, t_0)$$

Solve this by iteration, we have

$$\boxed{\begin{aligned} U(t, t_0) = 1 - i \int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \cdots \\ + (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1) H_I(t_2) \cdots H_I(t_n) + \cdots \end{aligned}}$$

Note that $H_I(t)$ does not commute with $H_I(t')$ at a different time. Thus the ordering of H's is very important.
It is a time ordered product! $t_1 < t_2 < \cdots < t_n$ as instructed by the integrals.

- physical meaning of U:
  If the system is known to be in state i at t_0, then the probability of finding the system in state f at some later time t is given by the modulus of the matrix element $\langle \Phi_f | U(t, t_0) \Phi_i \rangle$:

$$|\langle \Phi_f | U(t, t_0)\, \Phi_i \rangle|^2 = |U_{fi}(t, t_0)|^2$$

In physics, we usually do experiments by allowing the system to interact for a long time with respect to the micro-scale. Practically, the limit $t_0 \to -\infty$, $t \to \infty$ is more interesting. Thus we define the <u>S-matrix</u> (Scattering-matrix) by:

$$\boxed{\begin{aligned} S &= U(+\infty, -\infty) \\ S &= 1 - i \int_{-\infty}^{\infty} dt_1\, H_I(t_1) + \cdots + (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\, H_I(t_1) \cdots H_I(t_n) + \cdots \end{aligned}}$$

Remarks.

1.° Using the time ordering operator T, we can formally write $\mathcal{U}$ & S as

$$\mathcal{U}(t,t') = T \exp\left(-i\int_{t'}^{t} dt''\, H_I(t'')\right)$$

$$S = \mathcal{U}(+\infty, -\infty)$$

← T symbol acts on the whole expression. This means that after an expansion of the exponential in powers of H_I, all the fields φ have to be time ordered!

2.° $\mathcal{U}$ satisfies: $\begin{cases} \mathcal{U}(t_1,t_2)\,\mathcal{U}(t_2,t_3) = \mathcal{U}(t_1,t_3) \\ \mathcal{U}(t_1,t_2)^\dagger = \mathcal{U}(t_2,t_1) \\ \mathcal{U}(t_1,t_1) = 1 \end{cases}$

To show this, it is important to note that from the defining eqn.

$$i\frac{\partial}{\partial t}\mathcal{U}(t,t') = H_I(t)\,\mathcal{U}(t,t'),$$
$$\mathcal{U}(t,t) = 1$$

one can show that

$$\mathcal{U}(t,t') = e^{iH_0 t}\, e^{-iH(t-t')}\, e^{-iH_0 t'}$$

H, H_0 in Schrodinger picture on RHS

pf. $\mathcal{U}(t,t) = 1$ ✓

$$i\frac{\partial}{\partial t}\mathcal{U}(t,t') = -H_0\,\mathcal{U} + \underbrace{e^{iH_0 t}\, H\, \underbrace{e^{-iH(t-t')}\, e^{-iH_0 t'}}_{e^{-iH_0 t}\ \mathcal{U}(t,t')}}_{H_0 + \underbrace{e^{iH_0 t} H_I e^{-iH_0 t}}_{H_I \text{ in interacting picture}}}$$

$$= H_I^{(I)}\,\mathcal{U}$$

Dropping superscript, $i\frac{\partial}{\partial t}\mathcal{U} = H_I\,\mathcal{U}$ ✓ ,

3.° Thus $SS^\dagger = S^\dagger S = 1$

S is a unitarity operator.

In terms of states,

$$\sum_m S_{nm} S^*_{km} = \delta_{nk}$$

↑ Sum over all physical states

4° $H_I = \int d^3x\, \mathcal{H}_I = -\int d^3x\, \mathcal{L}_I$

$$S = T\exp\left[i\int d^4x\, \mathcal{L}_I\right]$$

↑ Covariant form of S-operator.

The expansion of S as power of H_I on p. 39 is a perturbative expansion of S in terms of the interaction.

In QED, H_I = electromagnetic interaction (more explicit later)
characterized by $\alpha = 1/137 \ll 1$

thus we can use this series rather efficiently to compute physical scattering processes.

For strong interaction, turns out $\alpha_{strong} \sim 1$
$\therefore$ perturbation theory is not useful.
Need new methods (difficult!)
Understanding of strong interaction is still poor!
QCD confinement is one of the fundamental problem for theoretical physics.

Scattering

Consider a scattering process $|i\rangle \to |f\rangle$

initial state $|i\rangle$ is prepared at $t_1 \to -\infty$

final state $|f\rangle$ is detected at $t_2 \to \infty$ ← time scale long enough to be considered as ∞ when compared to micro. scale of interaction.

The probability amp. is the overlap of the wavefun. $|f\rangle$ with $U(t_2, t_1)|i\rangle$

i.e. $\langle f| U(t_2, t_1) |i\rangle$.

since $t_1, t_2 \to -\infty, \infty$

$\therefore$ the probability amp is given by the S-matrix element $S_{fi} = \langle f| S |i\rangle$

• For real KG field,

using the perturbative expansion of S on p. 39, we need to compute correlators of the form:

$$\langle f| T\, \phi(x_1) \cdots \phi(x_n) \,| i\rangle$$

now $\begin{cases} |i\rangle = a^+_{k_1} a^+_{k_2} \cdots a^+_{k_n} |0\rangle \\ |f\rangle = a^+_{q_1} \cdots a^+_{q_m} |0\rangle \end{cases}$ scattering of n particles $\longrightarrow$ m particles

Thus we need to know how to compute VEV of form:

$$\langle 0| a\text{'s}\; T(\phi(x_1) \cdots \phi(x_n))\; a^+\text{'s} |0\rangle$$

To do this, we need the following Wick's Theorem which tells us how to write $T(\phi(x_1) \cdots \phi(x_n))$ in terms of simpler objects!

Wick's Theorem and Computation of $\langle 0| T\, \phi(x_1) \cdots \phi(x_n) |0\rangle$.

- In interaction picture, $\phi(x)$ is a free field, $|0\rangle$ is its fock vac.

We have

$\langle 0| \phi(x) |0\rangle = 0$

$\langle 0| T\ \phi(x)\ \phi(y) |0\rangle = i\,\Delta_F(x-y)$ p. 35

Since ϕ is free, can one relate $\langle 0| T(n \text{ fields}) |0\rangle$ with $\langle 0| T(2 \text{ fields}) |0\rangle$?

- Write $\phi = \phi^+ + \phi^-$

$$\phi^+ = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, a_p\, e^{-ipx} \qquad \text{the freq. part.}$$

$$\phi^- = \int \quad " \quad a_p^+\, e^{ipx} \qquad -\text{ve freq. part.}$$

$$\phi^+ |0\rangle = 0, \qquad \langle 0| \phi^- = 0$$

then for $x^0 > y^0$

$$\begin{aligned} T\,\phi(x)\,\phi(y) &= \phi^+(x)\,\phi^+(y) + \phi^+(x)\,\phi^-(y) + \phi^-(x)\,\phi^+(y) + \phi^-(x)\,\phi^-(y) \\ &= \phi^+(x)\,\phi^+(y) + \phi^-(y)\,\phi^+(x) + \quad " \quad + \quad " \\ &\quad + [\phi^+(x), \phi^-(y)] \end{aligned}$$

Note that in the first line, all a's are to the right of a^+'s.
Such a term (eg. $a_p^+ a_q^+ a_k a_\ell$) is <u>normal ordered</u>!
and has vanishing VEV!

Let us define the normal ordering symbol $N(\ldots)$ (: :, before)

e.g. $N(a_p a_k^+ a_q) \triangleq a_k^+ a_p a_q$

For $y^0 > x^0$, we will get the same four normal ordered terms, but the final commutator would be $[\phi^+(y), \phi^-(x)]$.

$\therefore$ Define the <u>contraction</u> of two fields by

$$\overbrace{\phi(x)\,\phi(y)} = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases} \qquad c\text{-numbers!}$$

This is exactly the Feynman propagator:

$$\overbracket{\phi(x)\,\phi(y)} = \Delta_F(x-y)$$

Then $$\boxed{T\left(\phi(x)\,\phi(y)\right) = N\left\{\phi(x)\,\phi(y) + \overbracket{\phi(x)\,\phi(y)}\right\}}$$

More general, Wicks Theorem:

$$\boxed{T\left(\phi(x_1)\cdots\phi(x_m)\right) = N\left\{\phi(x_1)\,\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}\right\}}$$

eg. $$T(\phi_1\phi_2\phi_3\phi_4) = N\left\{\begin{array}{l} \phi_1\phi_2\phi_3\phi_4 + \overbracket{12}\,34 + \overbracket{123}\,4 + \overbracket{1234} \\ + 1\,\overbracket{23}\,4 + 1\,\overbracket{234} + 12\,\overbracket{34} \\ + \overbracket{12}\,\overbracket{34} + (13)(24) + (14)(23) \end{array}\right\}$$

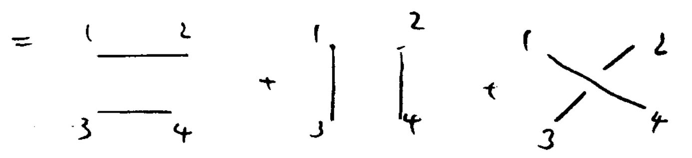
$N(\overbracket{\phi_1\phi_2\phi_3}\phi_4)$ means $N(\phi_2\phi_4)\cdot\Delta_F(x_1-x_3)$

Cor. One can express $\langle 0|T(\phi_1\cdots\phi_m)|0\rangle$ in terms of products of $\Delta_F(x,y)$.

$$\langle 0|T(\phi_1\cdots\phi_{2n+1})|0\rangle = 0$$

$$\langle 0|T(\phi_1\cdots\phi_{2n})|0\rangle = \underbrace{\Delta_F(x_1,x_2)\cdots\Delta_F(x_{2n-1},x_{2n})}_{n \text{ of them.}} + \cdots$$

eg. $$\langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \Delta_F(x_1,x_2)\,\Delta_F(x_3,x_4) + (13)(24) + (14)(23)$$



· Wick Thm. allows us to reduce Time ordered product to normal ordered product, which is particular convenient for evaluating VEV of time ordered product.

Feynman rules for KG real scalar field with $\lambda\phi^4$ interaction

- Consider KG real scalar with action

$$S = \int d^4x \ \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

- $$H = \int d^3x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right]$$

$$= H_0 + H_I$$

$$H_0 = \int d^3x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right]$$

$$H_I = \int d^3x \ \frac{\lambda}{4!}\phi^4 \qquad \lambda \text{ dimensionless constant}$$

We will consider $\lambda \ll 1$ s.t. we can think of H_I as a perturbation.

- Consider the process $p_1 + p_2 + p_3 + p_4 \to 0$

$$\text{Scattering matrix} = \langle 0 | S | p_1, p_2, p_3, p_4 \rangle = \underbrace{\langle 0 | p_1, p_2, p_3, p_4 \rangle}_{\phi} + \langle 0 | \frac{-i\lambda}{4!} T\left(\int d^4x \ \phi(x)\right) | p_1, p_2, p_3, p_4 \rangle + \ldots$$

This term does not correspond to any scattering !

For the next term, we use Wick theorem,

$$T\left(\int d^4x \ \phi^4(x)\right) = N\left(\int d^4x \ \phi^4(x) + \text{contractions}\right)$$

Since we are not taking expectation values between vaccum, the nonvanishing term may comes from 3 types of terms :

Ⓘ $\langle 0 | \ N(\phi\phi\phi\phi) \, a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger | 0 \rangle \qquad a_1^\dagger \hat{=} a^\dagger(p_1)$

Ⓘⓘ $\langle 0 | \ N(\phi\phi) \, \overbrace{\phi\phi} \, a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger | 0 \rangle$

Ⓘⓘⓘ $\langle 0 | \ \overbrace{\phi\phi} \ \overbrace{\phi\phi} \, a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger | 0 \rangle \qquad$ Schmatically

But obviously, Ⓘ & Ⓘ = 0

As for Ⓘ, $N(\phi\phi\phi\phi) = aaaa + \underbrace{a^3(a^+) + a^2(a^+)^2 + a(a^+)^3 + a^+a^+a^+a^+}_{\text{do not contribute}}$

$$Ⓘ = \langle 0|\, a\,a\,a\,a\;\; a^+_1 a^+_2 a^+_3 a^+_4\;|0\rangle$$

$$= \langle 0|\, a\,a\,a\,a\;\; a^+\, a^+\; a^+\, a^+\, |0\rangle \quad + \text{ other ways of contractions}$$

total $4! = 24$ ways, all give the same contributions.

$$\therefore \langle 0|\, \frac{-i\lambda}{4!}\int d^4x\; \phi^4(x)\;\; a^+_1 a^+_2 a^+_3 a^+_4\, |0\rangle$$

$$= \frac{-i\lambda}{4!}\int d^4x\; \frac{d^3q_1\cdots d^3q_4}{(2\pi)^3 2q_1^0 \cdots (2\pi)^3 2q_4^0}\cdot e^{-i(q_1+\cdots+q_4)x}\; \underbrace{\langle 0|\, a(q_1)\cdots a(q_4)\, a^+(p_1)\cdots a^+(p_4)\,|0\rangle}$$

$$\begin{cases} \left((2\pi)^3 2q_1^0\right)\cdots\left((2\pi)^3 2q_4^0\right)\, \delta^{(3)}(q_1-p_1)\cdots\delta^{(3)}(q_4-p_4) \\ + \text{ 23 other terms with different identification of } p\text{'s} \;\&\; q\text{'s} \end{cases}$$

$$= -\frac{i\lambda}{4!} \times 4! \times \int d^4x\;\; e^{-i(p_1+\cdots+p_4)x}$$

$$= -i\lambda\, (2\pi)^4\, \delta^{(4)}(\Sigma p)$$

$$\therefore\; S_{fi} = \underbrace{(2\pi)^4\,\delta^{(4)}(\Sigma p)}_{\substack{\text{universal} \\ \text{kinematical} \\ \text{factor}}} \left\{ \phi - i\lambda + o(\lambda^2) \right\},$$

In general,

we write $S_{fi} = (2\pi)^4\,\delta^{(4)}(P_f - P_i)\left\{\delta_{fi} + T_{fi}\right\}$ ← with $\delta^{(4)}(P_f - P_i)$ stripped off $\therefore$ finite!

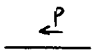
T_{fi} is called scattering amplitude.

We can represent diagrmatically the interaction as:

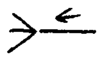
 $-i\lambda$

Without going to the details, and without proof, let us give the Feynman rules for the $\lambda\phi^4$ theory.

1.° 4-Momentum is conserved at each vertex
2.° loop momentum is integrated over $\int d^4p/(2\pi)^4$
} ← same for all QFT

3.° propagator  $= \dfrac{i}{p^2-m^2+i\varepsilon}$

4.° ϕ^4 vertex  $= -i\lambda$

5.° External scalar  $= 1$

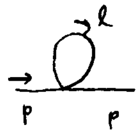
} Particular for ϕ^4.

- How to use these rules?
  - i. We write down all possible diagrams that can be constructed using these rules,
  - ii. At each order of λ, there is a finite number of diagrams one can draw
  - iii. Then we use the Feynman rules to compute each diagram as a multi-dim. integral.
  - iv. The sum of all these integrals give T_{fi}.

Eg. propagators $p \to p$

Tree level : $\longrightarrow \; = \dfrac{i}{p^2 - m^2 + i\varepsilon}$

one loop :



$$= \frac{-i\lambda}{4!} \int \frac{d^4 \ell}{(2\pi)^4} \cdot \frac{i}{\ell^2 - m^2 + i\varepsilon} \times S$$

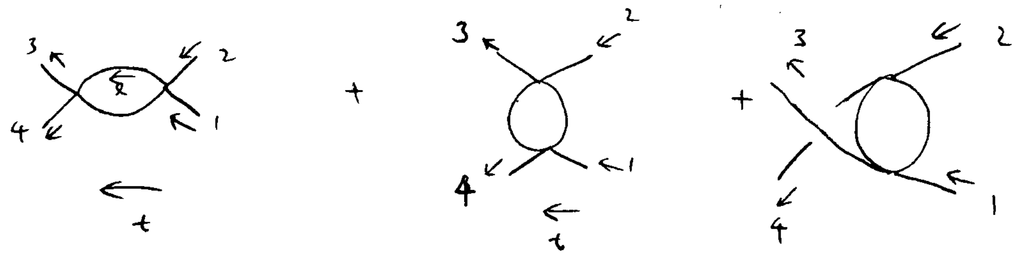
Symmetry factor
= # of ways one can contract
$\langle 0 | a\, T(\phi\phi\phi\phi)\, a^{+} | 0 \rangle$

$S = 12$ in this case.

$$= \frac{-i\lambda}{2} \int \frac{d^4 \ell}{(2\pi)^4} \frac{i}{\ell^2 - m^2 + i\varepsilon}$$

Eg. $2 \to 2$ scattering $p_1 + p_2 \to p_3 + p_4$

one loop



(a) (b) (c)

3 distinct contributions

$$(a) = \frac{(-i\lambda)^2}{2} \int \frac{d^4 \ell}{(2\pi)^4} \frac{i}{\ell^2 - m^2 + i\varepsilon} \frac{i}{(p_1 + p_2 - \ell)^2 - m^2 + i\varepsilon} ,$$

Check the symmetric factor $S = 12 \times 12 \times 2 = 4!\, 3!$

Feynman rule for QED

$$\mathcal{L} = \bar\psi (i\not\partial - m)\psi - \frac{1}{4} F_{\mu\nu}^2 - e\bar\psi\gamma^\mu\psi A_\mu$$

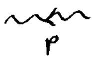
$\Big[\ \because\ \bar\psi i \not D \psi = \bar\psi i\not\partial\psi - e\bar\psi\gamma^\mu\psi A_\mu$

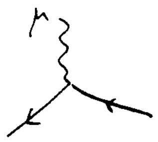
$D_\mu = \partial_\mu + ieA_\mu$

P.12 covariant derivative

1° + 2°

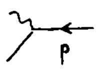
3° Dirac propagator : $\xleftarrow[p]{} \quad = \dfrac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$

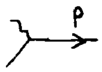
4° photon propagator :  $= \dfrac{-i g_{\mu\nu}}{p^2 + i\varepsilon}$ "Feynman gauge"



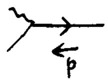
$= -ie\gamma^\mu$

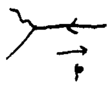
5° External fermions (e)

 $= u^s(p)$ (initial)

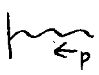
 $= \bar u^s(p)$ (final)

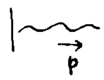
External anti-fermions (e^+)

 $\bar v^s(p)$ (initial)

 $v^s(p)$ (final)

External photons

 $\varepsilon_\mu(p)$ (initial)

 $\varepsilon_\mu^*(p)$ (final)

Fourier expansion of $A_\mu = \displaystyle\int \frac{d^3k}{(2\pi)^3\, 2k_0} \sum_{\lambda=1}^{2} \varepsilon_\mu^{(\lambda)} \left[a^{(\lambda)}(k)\, e^{-ikx} + a^{(\lambda)+} e^{ikx} \right]$

$\varepsilon_\mu^{(\lambda)}$ = polarization vector $\quad \lambda = 1, 2$

two polarization

6° Additional (-1) factor for a fermion loop.

A Real Example

Mott Scattering in QED

$$H_I = e\bar{\psi}\gamma_\mu\psi A_\mu$$

We will treat A_μ as a classical field (since we have not learnt how to quantize A_μ)

This is ok at the lowest order of approx.

Want to study scattering of relativistic electron in a Coulomb potential

$$\vec{A}=0 \quad , \quad A_0 = -eZ/4\pi r$$

This is a generalization of Rutherford scattering.

let initial momentum $=\vec{p}$, initial state $|i\rangle = b_\alpha^+|0\rangle\sqrt{m/EV}$

final momentum $=\vec{p}\,'$ final state $|f\rangle = b_{\alpha'}^{+}|0\rangle\sqrt{m/EV}$

$$\bar{\psi}\gamma_\mu\psi = \left(\bar{\psi}^{(+)}+\bar{\psi}^{(-)}\right)_i(\gamma_\mu)^i_{\ j}\left(\psi^{(+)}+\psi^{(-)}\right)^j$$

$$\psi = \int\frac{d^3k}{(2\pi)^3}\frac{m}{k_0}\left[\begin{array}{l} b_\alpha u^{(\alpha)}e^{-ikx} \\ +d_\alpha^+ v^{(\alpha)}e^{ikx}\end{array}\right]\begin{array}{l}\leftarrow\psi^{(+)}\\ \leftarrow\psi^{(-)}\end{array}$$

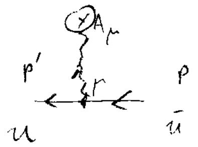
$$=(\gamma_\mu)^i_{\ j}\left[\bar{\psi}^{(+)}\psi^{(+)}+\bar{\psi}^{(+)}\psi^{(-)}+\bar{\psi}^{(-)}\psi^{(+)}+\bar{\psi}^{(-)}\psi^{(-)}\right]_i^{\ j}$$

$\uparrow$ only this term contribute in $\langle 0|\sim|0\rangle$

$$S_{fi}^{(1)} = -ie\langle f|\int d^4x\,\bar{\psi}\gamma^\mu\psi A_\mu|i\rangle$$

$$= -iem\int d^4x\,(\gamma^\mu)^i_{\ j}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\frac{m^2}{k_0k_0'}\cdot\bar{u}_i^{(\beta')}u_j^{(\beta)}\underbrace{\langle 0|b_{\alpha'}\,b_{\beta'}^+\,b_\beta\,b_\alpha^+|0\rangle}_{(2\pi)^6\frac{k_0k_0'}{m^2}\cdot\delta^{(3)}(k-p)\,\delta^{(3)}(k'-p')\,\delta_{\alpha\alpha'}\delta_{\beta\beta'}}e^{i(k'-k)\cdot x}\cdot A_\mu$$

$$= -ime\int d^4x\,\underbrace{\bar{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p)}_{x\text{-independent}}\cdot e^{i(p'-p)\cdot x}A_\mu(x)$$



$$= \bar{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p)\cdot -ime\int d^4x\,e^{i(p'-p)\cdot x}A_\mu(x)$$

In our case, only $A_0 \neq 0$,

$$\therefore \int d^4x\, e^{i(p'-p)\cdot x} A_0 = -\frac{Ze}{4\pi} \underbrace{\int dt\, e^{i(E'-E)t}}_{2\pi\delta(E'-E)} \underbrace{\int d^3x\, \frac{e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}}{r}}_{\frac{4\pi}{|\vec{p}-\vec{p}'|^2}} \qquad \int d^3x\, \frac{e^{i\vec{\alpha}\cdot\vec{r}}}{r} = \frac{4\pi}{|\vec{\alpha}|^2}$$

$$= -2\pi Ze\, \frac{\delta(E'-E)}{|\vec{p}-\vec{p}'|^2}$$

The transition rate is $|S^{(1)}_{fi}|^2$ divided by the time interval $t_f - t_i$.
We can get rid of one of the delta function by noticing that.

$$\delta(E-E') = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T/2}^{T/2} e^{i(E-E')t}\, dt$$

$$\therefore |2\pi\,\delta(E-E')|^2 = 2\pi T\, \delta(E-E')$$

let $t_f = T/2$, $t_i = -T/2$, $T \to \infty$

we see that $|S^{(1)}_{fi}|^2/T$ is indep. of T. Good!

- The differential cross section $d\Omega$ is equal to the

$$d\Omega = \frac{(\text{transition rate}) \times (\text{phase space volume})}{\text{incident flux density}}$$

$$= \frac{|S_{fi}|^2}{T} \cdot \frac{V\, d^3p'}{(2\pi)^3} \Big/ \left(\frac{|\vec{p}|}{EV}\right)$$

$$= \frac{Z^2e^4}{4\pi^2} \cdot \frac{m^2\, |\bar{u}\gamma^0 u|^2}{E|\vec{p}|\, |\vec{p}-\vec{p}'|^4}\, \delta(E-E')\, d^3p'$$

$\vec{p} = \gamma m \vec{v}$
$E = \gamma m c^2$
$\frac{\vec{p}}{EV} = \frac{|\vec{v}|}{V}$ = flux density

using $d^3p'\, \delta(E'-E) = d|p'| \cdot |p'|^2 \cdot \delta(E'-E) \cdot d\Omega$

integrate over final energy E', we get $\int_0^\infty d^3p'\, \delta(E'-E) = d\Omega \frac{|\vec{p}|^2}{\partial E/\partial |p|} = |\vec{p}|\, E\, d\Omega$

$+ |\vec{p}'| = |\vec{p}|$

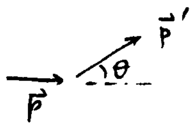
$$\therefore \frac{d\sigma}{d\Omega} = \frac{Z^2 e^4 m^2}{4\pi^2} \cdot \frac{|\bar{u}\gamma^0 u|^2}{|\vec{p}-\vec{p}\,'|^4}$$

$$= \frac{Z^2 e^4 m^2}{4^3 \pi^2 |p|^4 \sin^4\theta/2} \cdot |\bar{u}\gamma^0 u|^2$$

$$= \frac{1}{4}(Z\alpha)^2 \frac{m^2}{|p|^4 \sin^4\theta/2} \cdot |\bar{u}\gamma^0 u|^2$$

$|\vec{p}| = |\vec{p}\,'|$

$|\vec{p}-\vec{p}\,'|^2 = 4|p|^2 \sin^2\theta/2$



$\alpha = e^2/4\pi$

Spin sum

The problem now reduced to calculating $|\bar{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p)|^2$

For a particular transition from a specific spin state to some other spin state, we need to use the explicit form of u.

Often, we don't measure spin orientations in expt. → sum over final spin α' $\sum_{\alpha'}$

Further we may assume incident electrons beam is unpolarized → average over initial spin $\frac{1}{2}\sum_\alpha$

Therefore we are interested in,

$$\frac{1}{2}\sum_\alpha \sum_{\alpha'} \underbrace{\bar{u}_i^{(\alpha')}(p')}(\gamma^0)_{ij} \underbrace{u_j^{(\alpha)}(p) \quad \bar{u}_k^{(\alpha)}(p)}(\gamma^0)_{k\ell} \underbrace{u_\ell^{(\alpha')}(p')}$$

$$\left[\left(\bar{u}^{(\alpha')}(p')\gamma_0 u^{(\alpha)}(p)\right)^* = \bar{u}^{(\alpha)}(p)\gamma_0 u^{(\alpha')}(p') \right.$$

$$= \frac{1}{2}\left(\frac{\not{p}'+m}{2m}\right)_{\ell i} \left(\frac{\not{p}+m}{2m}\right)_{jk} (\gamma^0)_{ij} (\gamma^0)_{k\ell}$$

$$= \frac{1}{8m^2} \mathrm{Tr}\left[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right]$$

• Need :

$$\begin{cases} \mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) \\ \mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha) = 0 \\ \mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta_{\mu\nu} \end{cases}$$

← very useful!

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

$\eta^{\mu\nu} = (1, -1, -1, -1)$

Pf. $\mathrm{Tr}\,\gamma^\mu\gamma^\nu + \mathrm{Tr}\,\gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\cdot \underbrace{\mathrm{Tr}\,\mathbb{1}_{4\times4}}_{=4}$

$\Rightarrow \mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$ //

$$\begin{aligned} \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha) &= \mathrm{Tr}[(\gamma^5)^2\gamma^\mu\gamma^\nu\gamma^\alpha] \\ &= -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^5) \\ &= -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha) \end{aligned}$$

• $(\gamma^5)^2 = 1$

• γ^5 anti commute with γ^μ's

$\Rightarrow \mathrm{Tr}(\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}}_{\text{odd number of }\gamma\text{'s}}) = 0$

$$\begin{aligned} \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) &= -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta) + 2\eta^{\mu\nu}\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta) \\ &= \mathrm{Tr}(\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta) + 8(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta}) \\ &= -\mathrm{Tr}(\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu) + 8(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) \end{aligned}$$

$\Rightarrow$ result ! //

$$\therefore \frac{1}{2}\sum_{\alpha}\sum_{\alpha'} |\bar{u}\gamma_0 u|^2$$

$$= \frac{1}{8m^2}\left[\underbrace{\mathrm{Tr}(\not{p}'\gamma^0\not{p}\gamma^0)}_{\substack{8E^2-4p\cdot p' \\ =8E^2-4E^2+4\vec{p}\cdot\vec{p}\,'}} + 4m^2\right]$$

$\vec{p}\cdot\vec{p}\,' = |\vec{p}|^2\cos\theta = |\vec{p}|^2(1-2\sin^2\theta/2)$

$$= \frac{1}{m^2}\left(E^2 - |\vec{p}|^2\sin^2\theta/2\right)$$

$$= \frac{E^2}{m^2}\left(1-\beta^2\sin^2\theta/2\right)$$

$\beta = |\vec{p}|/E$

$$\therefore \boxed{\overline{\left(\frac{d\sigma}{d\Omega}\right)} = \frac{(Z\alpha)^2}{4}\frac{E^2}{|\vec{p}|^4\sin^4\theta/2}\left(1-\beta^2\sin^2\theta/2\right)}$$

At low speed, $\beta \sim 0$

$|\vec{p}| \simeq mv$

$E \simeq m$,

$$\overline{\left(\frac{d\sigma}{d\Omega}\right)} = \frac{(Z\alpha)^2}{4}\frac{1}{m^2v^4\sin^4\theta/2}$$

Rutherford Cross section

Susy

In the 60's, growing awareness of the significance of internal symmetries
eg. isospin $SU(2)$.
physicists started to ask whether it is possible to find a symm. which can combine space-time Poincare symm. and internal symm. in a nontrivial way.

Coleman & Mandula Theorem (1967)

They showed that, on very general assumptions, any Lie gp. which contain the Poincare gp. P and an internal symm. group. G must be a direct product of P & G.

ie P & G commute.

trivial mixing!

Equivalent, if one use a Lie group that contain an internal group which mixed nontrivially with the Poincare gp., then the S-matrix would be identically zero!

ie. trivial physics.

There is however exception to the no-go theorem.
Coleman-Mandula considered internal symm described by Lie alg.

ie. $[T^i, T^j] = f^{ijk}\, T^K$ T^i = generators of Lie alg.

Commutators.

Nontrivial mixing of spacetime symm and internal symm. is possible if the internal symm. was generated by generators that satisfy the relations

$$\{Q_\alpha, Q_\beta\} \hat{=} Q_\alpha Q_\beta + Q_\beta Q_\alpha$$

anti commutators

= some other generators.

The generators Q_α generate a new symmetry (internal) called Super symmetry.

Q_α is called supercharges

- The Poincaré symm. generated by generators P_μ, $J_{\mu\nu}$

$\uparrow$ translations $\quad \nwarrow$ Lorentz rotations

$$[P_\mu, P_\nu] = 0$$
$$[P_\mu, J_{\nu\lambda}] = \eta_{\mu\nu} P_\lambda - \eta_{\mu\lambda} P_\nu$$
$$[J_{\mu\nu}, J_{\alpha\beta}] = \eta_{\nu\alpha} J_{\mu\beta} + \eta_{\mu\beta} J_{\nu\alpha} - \eta_{\mu\alpha} J_{\nu\beta} - \eta_{\nu\beta} J_{\mu\alpha}$$

and the supersymmetry generators Q_α

mixed in the following way:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu)_{\alpha\beta} P_\mu$$
$$[Q_\alpha, J_{\mu\nu}] = \tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta$$
$$[Q_\alpha, P_\mu] = 0$$

In other words, we can introduce a Z_2 grading:

odd: Q_α

even: P_μ, $J_{\mu\nu}$

and the susy alg. has the structure

$$\{\text{odd}, \text{odd}\} = \text{even}$$
$$[\text{odd}, \text{even}] = \text{odd}$$
$$[\text{even}, \text{even}] = \text{even}$$

Wess Zumino model

· Now I want to show you the simplest field theory with supersymmetry.
The Wess Zumino model is defined by the Lagrangian density

$$L = -\frac{1}{2}(\partial_a A)^2 - \frac{1}{2}(\partial_a B)^2 - \frac{i}{2}\bar{\psi}\gamma^a\partial_a\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2$$

A, B, F, G, ψ real $\qquad \partial_a = \frac{\partial}{\partial x^a}$

EoM:
$$\begin{cases} \Box A = 0 \\ \Box B = 0 \\ \gamma^a\partial_a\psi = 0 \\ F = 0 \quad \leftarrow \text{ nonpropagating ! auxillary} \\ G = 0 \quad \leftarrow \end{cases} \qquad \text{— ①}$$

SUSY transformation is defined by:

$$\begin{cases} \delta A = i\bar{\alpha}\psi \\ \delta B = i\bar{\alpha}\gamma_5\psi \\ \delta\psi = \left[\gamma^a\partial_a(A + \gamma_5 B) + F + \gamma_5 G\right]\alpha \\ \delta F = i\bar{\alpha}\gamma^a\partial_a\psi \\ \delta G = i\bar{\alpha}\gamma_5\gamma^a\partial_a\psi \end{cases} \qquad \text{— ②}$$

α infinitesimal / constant / anticommuting / spinorial parameter.

· ψ Majorana spinor i.e. ψ has real components.
It is convenient to choose the Majorana reps. of Gamma matrix.

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab},$$
γ^a : real 4×4 matrices (Majorana reps.)

lemma For any two real anticommuting spinors α_1, α_2
$$\bar{\alpha}_1\Gamma\alpha_2 = \bar{\alpha}_2\widetilde{\Gamma}\alpha_1 \qquad \bar{\alpha} = \alpha^T\gamma^0.$$

where $\widetilde{\Gamma} \hat{=} -\gamma_0\Gamma^T\gamma_0$

In particular, $\widetilde{\gamma_A} = \begin{cases} -\gamma_A & \text{for } \gamma_a \\ +\gamma_A & \phantom{\text{for }} \gamma_5 \end{cases}$.

Using this, one can show that

$$\delta L = \frac{i}{2}\partial^a\left[\bar{\alpha}\gamma_a\{F - \gamma_5 G - \gamma\cdot\partial(A+\gamma_5 B)\}\psi\right]$$

= total derivative

$$\therefore \delta S = \delta\int d^4x\, L = 0$$

$\therefore$ The transformation ① is a symmetry of the action!

Note that :

1° Supersymmetry transforms bosons $\leftrightarrow$ fermions. A symmetry that identifies bosons and fermions! Amazing!

2° In a supersymmetric theory, always equal number of bosonic & fermionic degree of freedom.

eg. in WZ model,

$$\left.\begin{array}{lll} A, B & \rightarrow & 2 \text{ real bosons} \\ F, G & \rightarrow & \phi \quad (\text{auxillary}) \end{array}\right\} \quad \#_B = 2$$

ψ Majorana $\rightarrow$ 2 real components $\qquad \#_F = 2$

3° The fields (A, B, F, G, ψ) form a multiplet called Supermultiplet.
EoM transformed to each other under susy.

e.g. $\delta(\Box A) = \Box(\delta A) = i\bar{\alpha}\Box\psi = i\bar{\alpha}\gamma\cdot\partial(\gamma\cdot\partial\psi)$

$$\delta(\gamma\cdot\partial\psi) = \gamma\cdot\partial\left[\gamma\cdot\partial(A+\gamma_5 B) + F + \gamma_5 G\right]\alpha ,$$

$$= \left[\Box(A+\gamma_5 B) + \gamma\cdot\partial(F + \gamma_5 G)\right]\alpha$$

ie. $(F, G, \Box A, \Box B, \gamma\cdot\partial\psi)$ also form a supermultiplet.

Susy alg.

Consider the commutators of supersymmetry transformations

$$\delta_2\delta_1 A = i\bar{\alpha}_1\, \delta_2\psi = i\bar{\alpha}_1 \left[\gamma\cdot\partial(A+\gamma_5 B) + F + \gamma_5 G\right]\alpha_2$$

$$\begin{cases} \bar{\alpha}_1\alpha_2 = \bar{\alpha}_2\alpha_1 \\ \bar{\alpha}_1\gamma_5\alpha_2 = \bar{\alpha}_2\gamma_5\alpha_1 \\ \bar{\alpha}_1\gamma_5\gamma_a\alpha_2 = \bar{\alpha}_2\gamma_5\gamma_a\alpha_1 \\ \bar{\alpha}_1\gamma_a\alpha_2 = -\bar{\alpha}_2\gamma_a\alpha_1 \end{cases}$$

$$\therefore (\delta_2\delta_1 - \delta_1\delta_2)A = 2i\bar{\alpha}_1\gamma^a\alpha_2\,\partial_a A = \xi^a\partial_a A$$

$$\xi^a = 2i\bar{\alpha}_1\gamma^a\alpha_2$$

This is the same for all the other fields in the supermultiplet:

$$\delta_2\delta_1 - \delta_1\delta_2 = 2i\bar{\alpha}_1\gamma^a\alpha_2\,\partial_a =$$

If we remember $P_a = i\partial_a$ is the translation generators

then $\delta_2\delta_1 - \delta_1\delta_2 = 2\xi^a P_a$

Commutator of two Susy transformations is a translation.

Recall: anticommutator of two super charges (odd) is equal to P_a

Note that Q_α is odd

but $\delta_\epsilon = \bar{\epsilon}^\alpha Q_\alpha$ is even,

$\bar{\epsilon}^\alpha$ odd, Q_α odd

$\bar{\epsilon}^\alpha$: anticommuting parameter

Susy field theories have better ultraviolet properties.

Nonrenormalization Theorem

→ That's another course! Have fun!