Physics is a studies of Symmetries

eg SR →

GR → General covariance

EM → U(1) , W → SU(2) , S → SU(3)

Worldsheet → Conformal inv.

$$\text{Symmetry} \begin{array}{l} \rightarrow \text{Conservation law} \\ \searrow \text{interaction} \end{array}$$

How is Symmetry represented in Field theory ?

$$\text{Field theory} \begin{array}{l} \diagup \text{Hamiltonian} \\ \diagdown \text{Lagrangian formulation} \xleftarrow{\quad} \text{focused on this} \end{array}$$

↓

<u>Noether theorem</u>

$$S = \int \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))\, d^4x$$

$$\delta S = 0 \Rightarrow \text{EOM} \quad \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} - \frac{\delta \mathcal{L}}{\delta \phi_i} = 0$$

<u>Thm</u> (Noether 1918)

Any continuous Global Symmetry of S (ie leaves S inv.) implies the existence of a conserved current, $\partial_\mu J^\mu(x) = 0$

<u>Cor</u> $Q(t) \triangleq \int d^3x\, J_0(x)$ is a conserved quantity, $\frac{dQ}{dt} = 0$

it is called the "charge" of the symmetry.

<u>(pf of Cor)</u> $$\frac{dQ}{dt} = \int d^3x\, \partial_0 J_0 = -\int d^3x\, \underset{\substack{\uparrow \\ \text{Total derivative}}}{\partial_i} J_i = 0 \xleftarrow{\quad} \text{However exception eg instanton}$$

(pf Thm) $$\delta\mathcal{L} = \underbrace{\frac{\delta\mathcal{L}}{\delta\phi_i}}_{\substack{\| \,\text{EoM} \\ \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}}} \delta\phi_i + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)} \underbrace{\delta(\partial_\mu\phi_i)}_{\substack{\| \\ \partial_\mu(\delta\phi_i)}}$$

$$= \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\delta\phi_i\right]$$

S inv. $\Rightarrow \delta\mathcal{L} = \partial_\mu K^\mu$ for some K^μ

$\therefore \partial_\mu \tilde{J}^\mu = 0$ with $\tilde{J}^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\delta\phi_i - K^\mu$

eg. $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$

finite $\begin{cases} \psi \to g^{-1}\psi \\ \bar{\psi} \to \bar{\psi} g \end{cases}$ is a global Symm, where $g = e^{\theta^a T^a}$

infinitesmal $\begin{cases} \delta_\theta\psi = -\theta\psi \\ \delta_\theta\bar{\psi} = \bar{\psi}\theta \end{cases}$

Easy to show that $\delta_\theta\mathcal{L} = 0$

$\therefore \tilde{J}^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\psi = -\underset{\uparrow \text{arb.}}{\theta^a}(i\bar{\psi}\gamma^\mu T^a\psi)$

we get a set of conserved current : $\partial_\mu J^\mu_a = 0$

$J^\mu_a = i\bar{\psi}\gamma^\mu T_a\psi$

Also, another global Symm — vector current

$\begin{cases} \psi \to e^{i\alpha\gamma_5}\psi \\ \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5} \end{cases}$ or $\begin{cases} \delta\psi = i\alpha\gamma_5\psi \\ \delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5 \end{cases}$

$\tilde{J}^\mu_5 = -\alpha(\bar{\psi}\gamma^\mu\gamma_5\psi)$

$\Rightarrow J^\mu_5 = \bar{\psi}\gamma^\mu\gamma_5\psi$ Axial vector Current

rmk. $K^\mu \neq 0$ for susy

ie susy charge Q has a contribution from K^μ.

"Charge" as Generator of Symmetry

What is the physical meaning of Q ?

One can show that $\delta\phi_i = i[Q, \phi_i]$

pf. $Q = \int d^3x\, J_0 = \int d^3x \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_0\phi_i)}}_{\shortparallel}\delta\phi_i$ $K_\mu = 0$ for simplicity

π_i Canonical momentum conjugate to ϕ_i

i.e. $[\pi_i(t,\vec{x}), \phi_j(t,\vec{y})] = -i\delta_{ij}\delta(\vec{x}-\vec{y})$

$\therefore [Q, \phi_i] = -i\delta\phi_i$ //

we can integrate and get the finite form,

$$\phi_i \to \phi_i' = e^{iQ}\phi_i e^{-iQ} = U\phi_i U^{-1}$$

Thus $U = e^{iQ}$ generate the transformation

and Q is the generator of the symmetry.

Why is Q called charge ?

In the simplest case (eg. $U(1)$ free fermion model we considered above)

$Q = \int d^3x\, \bar\psi\gamma^0\psi = \int d^3x\, \psi^\dagger\psi$ is the charge operator.

In more complicated examples, we can have many Q's and they are not invariant, thus Q are not physical observables. Nevertheless, we continuous to call them charges.

A more complicated example is the sigma model,

$$\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$$

ϕ_i form the local coordinates of a manifold M

M is acted on by a symmetry group G

$$G : M \to M$$

$$g : \phi_i \mapsto \phi_i'$$

$$g = e^{iT^a\varepsilon^a} \qquad [T^a, T^b] = if^{abc}T^c$$

i.e. $\delta\phi_i(x) = i\varepsilon^a T^a_{ij}\phi_j(x)$

$$\therefore \tilde{J}_\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)} i\varepsilon^a T^a_{ij}\phi_j(x) = \varepsilon^a J^a_\mu$$

$$J^a_\mu(x) = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)} i t^a_{ij}\phi_j$$

In particular, $J^a_0 = i\Pi_i t^a_{ij}\phi_j$ $\qquad \sim \vec{L} = \vec{p}\times\vec{x}$

This is similar to angular momentum $\vec{L} = \vec{p}\times\vec{x}$

Since $[L, L] = i\varepsilon L$

suggest to look for nontrivial alg. for J^a_0 or Q^a's.

In fact, easy to show

$$\boxed{[Q^a(t), Q^b(t)] = if^{abc}Q^c(t)}$$

This is called the charge algebra.

One can also calculate the CR. for the currents.

For example, for QED, we have

$$[J_0(t,\vec{x}), J^5_\mu(t,\vec{y})] = 0 \qquad \text{current algebra}$$

$$[J_0(t,\vec{x}), J_\mu(t,\vec{y})] = 0$$

Why are we interested in these currents?

Because they are just the <u>physical currents</u> which participate in <u>interactions.</u>

For example, electrons constitute the EM current J^μ and couple to EM field A^μ by the minimal coupling $\int J_\mu A^\mu$

Similarly, matters are represented by QF $\Phi_i(x)$ and they form the current J_μ (Noether construction).

These current couple to gauge fields thr. $J_\mu^a A_\mu^a$

Also, they can interact themselves thr. current-current interaction

$$\mathcal{L}' = J_\lambda^\dagger J^\lambda$$

(In String theory, this is called Sugawara construction

$T(z) = J J = \sum L_n z^{-n-2}$

The point is that L_n then satisfies Virasoro alg.

Nature is not perfect.

Symmetry are often broken.

For example, 1° it can be broken by <u>explicit Symm breaking</u> term $\mathcal{L}_1$ in the Lagrangian. of course, $\mathcal{L}_1 << \mathcal{L}$, Otherwise, there is not even a sense to talk about approx symm.

2° <u>SSB</u> – The action is symmetric.

But the ground state take a preferred configuration and breaks the symm.

– The symm. can be a local or a global one

If it is global → Goldstone bosons (massless, spin 0 particles)

If it is local → Goldstone bosons eaten up by massless gauge fields, A_μ becomes massive.

e.g. In standard model,.

$W^\pm, Z^0 \sim 100$ GeV

3° Quantum mechanical effects

The symmetry is said be broken by an <u>anomaly</u>.

often, they appears in model with <u>fermions</u>.

(Exception, e.g. self dual Tensor fields → gravitational anomaly)

Therefore we will follow history and start with these models

Abelian Anomaly

- Consider the FT with ψ massless Dirac fermion

  A_μ gauge field $A_\mu = A_\mu^a T^a$

  ↑ Lie alg $\mathfrak{g}$ antihermitian

Lagrangian (density) $\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + A_\mu)\psi$

$\mathcal{L}$ is invariant under 1° local gauge transformation

$$\psi \to g^{-1}\psi$$

$$A_\mu \to g^{-1}(A_\mu + \partial_\mu) g \quad , \quad g(x) = e^{\alpha^a(x) T^a}$$

2° global symmetry $\alpha =$ constant

$$\psi \to e^{i\gamma_5 \alpha}\psi$$

$$\bar{\psi} \to \bar{\psi} e^{i\gamma_5\alpha}$$

- $\gamma_5 \Rightarrow$ Even dim.
- $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$

This is called chiral symmetry

$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ chiral current

properties: 1/ Define $Q \triangleq \int d^3x\, j_5^0$

Show that $\delta_\alpha \psi = c\,\alpha\,[Q, \psi]$

What is the value of c?

2/ $\partial_\mu j_5^\mu = 0$ using EoM

- Turns out in the QFT, $\partial_\mu j_5^\mu \neq 0$

Adler 1969

Bell + Jackiw

↑ Computing feynman diagram

$$\boxed{\begin{aligned} \partial_\mu j_5^\mu &= \frac{1}{16\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\, tr(F_{\alpha\beta}F_{\gamma\delta}) \qquad \text{ABJ anomaly} \\ &= \frac{1}{4\pi^2}\, tr\left[\varepsilon^{\alpha\beta\gamma\delta}\,\partial_\alpha\left(A_\beta\partial_\gamma A_\delta + \frac{2}{3}A_\beta A_\gamma A_\delta\right)\right] \end{aligned}}$$

Nonabelian Anomaly

Consider ψ Weyl fermion $\Rightarrow$ Even dim.

$$A_\mu = A_\mu^a T^a \qquad \text{as before}$$

$$\mathcal{L} = \bar{\psi} i \not{D} \psi \qquad\qquad D_\mu = \partial_\mu + A_\mu$$

$\mathcal{L}$ is inv. under gauge symmetry, No more chiral invariance.

symmetry current = non. Abelian current

$$j^{\mu a} = \bar{\psi} \gamma^\mu T^a \psi$$

$$(D_\mu j^\mu)^a \triangleq \partial_\mu j^{\mu a} + f^{abc} A_\mu^b j^{\mu c} \qquad \text{Covariant divergence}$$

$$\boxed{(D_\mu j^\mu)^a = \frac{1}{24\pi^2} \operatorname{tr}\left[T^a \partial_\alpha \varepsilon^{\alpha\beta\gamma\delta} (A_\beta \partial_\gamma A_\delta + \frac{1}{2} A_\beta A_\gamma A_\delta)\right]}$$

Bardeen (1969)

Gross + Jackiw (1972)

Remarks 1° Anomaly $\mathcal{A}$ = divergence of current $\neq 0$ only possible for $D = \text{even}$

Chiral Anomaly	Abelian	Nonabelian	
1° reps	Singlet	Adjoint	← WZ consistency condition
2° Total div. ?	Y	Y	
3° coeff.	$\frac{1}{4\pi^2}, \frac{2}{3}$	$\frac{1}{24\pi^2}, \frac{1}{2}$	
4° Bad ?	No	Yes	

2° Here "QFT" means we quantize and integrate out the matter (fermion in the above). The gauge field is classical and is often refer to as $\underset{\sim}{\text{external}}$.

3° Abelian/Nonabelian refers to the reps. of current, not the gauge group.

Effective Action and Anomaly (Abelian or Nonabelian)

- $S = \int -\frac{1}{4}\mathrm{tr}F^2 + i\bar\psi \not{D}\psi$

The Quantum eff. action can be computed from Z, which can be computed in stages :

$$Z = \int \mathcal{D}A\, \mathcal{D}\psi\, e^{-S}$$

$$= \int \mathcal{D}A\, e^{-\mathrm{tr}F^2} \underbrace{\int \mathcal{D}\psi\, e^{-\int i\bar\psi \not{D}\psi}}_{\substack{\parallel \\ e^{-\Gamma[A]}}}$$

$\longleftarrow$ integrate over ψ

$\nwarrow$ Quantum eff action

$$= \int \mathcal{D}A\, e^{-\mathrm{tr}F^2}\, e^{-\Gamma[A]}$$

If $\delta\Gamma[A] \neq 0$, then Z is not invariant.

$\Rightarrow$ we have an anomaly.

- Reason :

$$e^{-\Gamma[A]} = \int \mathcal{D}\psi\, e^{-i\int \bar\psi \not{D}\psi}$$

Under $A_\mu \to A_\mu + D_\mu\alpha$, $\quad D_\mu\alpha = \partial_\mu\alpha + [A_\mu, \alpha]$

$$e^{-\Gamma[A+D_\mu\alpha]} - e^{-\Gamma[A]} = \int \mathcal{D}\psi\, e^{-i\int\bar\psi\gamma^\mu(\partial_\mu + A_\mu + D_\mu\alpha)\psi} - e^{-i\int\bar\psi\gamma^\mu(\partial_\mu + A_\mu)\psi}$$

$$e^{-\Gamma[A]}\left(e^{-\Gamma[A+D_\mu\alpha]+\Gamma[A]} - 1\right) = \int \mathcal{D}\psi\, e^{-i\int\bar\psi\gamma^\mu D_\mu\psi} \cdot -i\int\bar\psi\gamma^\mu D_\mu\alpha\,\psi$$

$$\Gamma[A+D_\mu\alpha] = \Gamma[A] + \int dx\, \mathrm{tr}\left(D_\mu\alpha\, \frac{\delta\Gamma}{\delta A_\mu}\right)$$

$$= \Gamma[A] - \int dx\, \mathrm{tr}\left(\alpha\, D_\mu \frac{\delta\Gamma}{\delta A_\mu}\right)$$

α arbitrary $\Rightarrow$ $D_\mu\left(\frac{\delta\Gamma}{\delta A_\mu}\right) = \langle D_\mu j^\mu\rangle \qquad j^{\mu a} = -i\bar\psi\gamma^\mu T^a\psi$

$$\Gamma[A+D_\mu\alpha] - \Gamma[A] = -\int dx\, \mathrm{tr}(\alpha \langle D_\mu j^\mu\rangle)$$

ie $\delta\Gamma \neq 0 \quad \Leftrightarrow \quad \langle D_\mu j^\mu\rangle \neq 0$

Computing Abelian Anomaly (Perturbative)

Consider QED as an example, $\mathcal{L} = \bar{\psi}\gamma^\mu(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$$J^\mu = \bar{\psi}\gamma^\mu\psi$$

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

Classically, $\partial_\mu J^\mu = 0$

$\partial_\mu J_5^\mu = 2imP$ where $P = \bar{\psi}\gamma_5\psi$

as a consequence of EOM

$$\begin{cases} i\not{\partial}\psi = m\psi \\ \partial_\nu F^{\mu\nu} = J^\mu \end{cases}$$

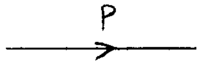
Below, we will compute the anomaly in 2 different ways :

- Feynman diagrams calculation
- Functional approach

Quantization: $\{\psi(t,x)_\beta, \bar{\psi}(t,y)^\alpha\} = (\gamma_0)_\beta{}^\alpha\,\delta^3(x-y)$

$$\langle 0|T\,\psi_\beta(y)\,\psi^\alpha(x)|0\rangle = i\,S_F(x-y)_\beta{}^\alpha$$

$$= \delta^\alpha_\beta \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{i}{\not{p}-m}$$



$$= \frac{i}{\not{p}-m}$$

Perturbative calculations

Consider the 3-pt. functions

$$T_{\mu\nu\lambda}(k_1, k_2, q) = i\int d^4x_1\, d^4x_2 \langle 0 | T\, J_\mu(x_1)\, J_\nu(x_2)\, J_\lambda^5(0) | 0\rangle\, e^{ik_1x_1 + ik_2x_2}$$

$$T_{\mu\nu}(k_1, k_2, q) = i\int d^4x_1\, d^4x_2 \langle 0 | T\, J_\mu(x_1)\, J_\nu(x_2)\, P(0)\, | 0\rangle \quad ''$$

where $q = k_1 + k_2$

Since $\partial_x^\mu \left(T\, J_\mu(x)\, \Omega(y)\right) = T\left(\partial^\mu J_\mu(x)\, \Omega(y)\right) + [J_0(x), \Omega(y)]\, \delta(x_0 - y_0)$

pf $\quad T\, J_\mu(x)\, \Omega(y) = J_\mu(x)\, \Omega(y)\, \theta(x_0 - y_0) + \Omega(y)\, J_\mu(x)\, \theta(y_0 - x_0)$

$\quad \partial_x^0\, \theta(x_0 - y_0) = \delta(x_0 - y_0)$ //

and $[J_0(t, x), J_\mu^5(t, y)] = 0$

$[J_0(t, x), J_\mu(t, y)] = 0$

$$\therefore \boxed{\begin{aligned} k_1^\mu T_{\mu\nu\lambda} &= k_2^\nu T_{\mu\nu\lambda} = 0 \\ q^\lambda T_{\mu\nu\lambda} &= 2m\, T_{\mu\nu} \end{aligned}}$$

Vector Ward identity

Axial vector Ward identity

However, if $\partial_\mu J_\mu \neq 0$ or $\partial_\mu J_\mu^5 \neq 0$, these Ward id. can be violated and are called anomalous Ward identity.

- The reason that the quantized current need not satisfy the classical conservation law is because the definition involve a product of fields at the same spacetime point. This is singular and must be regulated carefully.
- The regularization may spoil the classical symm.

  If there is no way to regulate the theory such that the classical symmetry is preserved, then we have an anomaly.

Triangular Diagrams

$$\langle 0|T J_\mu J_\nu J^5_\lambda|0\rangle = \langle 0|T\,\bar\psi\gamma_\mu\psi\;\bar\psi\gamma_\nu\psi\;\bar\psi\gamma_\lambda\gamma^5\psi|0\rangle$$

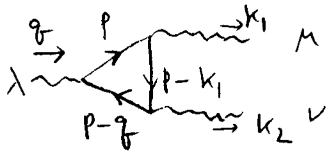
$$+\langle 0|T\,\bar\psi\gamma_\mu\psi\;\bar\psi\gamma_\nu\psi\;\bar\psi\gamma_\lambda\gamma^5\psi|0\rangle$$

To the lowest order in perturbation, ψ is free and we have

$$T_{\mu\nu\lambda} = i\int\frac{d^4p}{(2\pi)^4}(-1)\left\{\begin{array}{l}\mathrm{tr}\left[\dfrac{i}{\not{p}-m}\gamma_\lambda\gamma_5\dfrac{i}{\not{p}-\not{q}-m}\gamma_\nu\dfrac{i}{\not{p}-\not{k}_1-m}\gamma_\mu\right]\\ +\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}\end{array}\right\}$$

$$T_{\mu\nu} = i\int\frac{d^4p}{(2\pi)^4}(-1)\left\{\begin{array}{l}\mathrm{tr}\left[\dfrac{i}{\not{p}-m}\gamma_5\dfrac{i}{\not{p}-\not{q}-m}\gamma_\nu\dfrac{i}{\not{p}-\not{k}_1-m}\gamma_\mu\right]\\ +\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}\end{array}\right\}$$

$T_{\mu\nu\lambda}$ =



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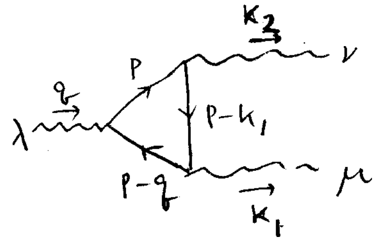
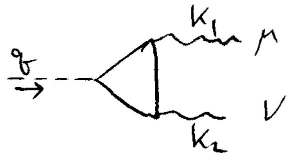


$T_{\mu\nu}$ =



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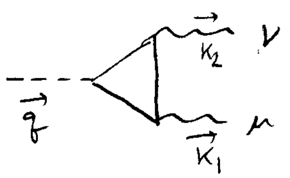


Now, $q^\lambda T_{\mu\nu\lambda} \sim \mathrm{tr}\left[\dfrac{i}{\not{p}-m}\underbrace{\not{q}\gamma_5}_{\parallel}\dfrac{i}{\not{p}-\not{q}-m}\gamma_\nu\dfrac{i}{\not{p}-\not{k}_1-m}\gamma_\mu\right] + \cdots$

$$\gamma_5(\not{p}-\not{q}-m)+(\not{p}-m)\gamma_5+2m\gamma_5$$

$$\therefore\ q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + \Delta_{\mu\nu}$$

$$\Delta_{\mu\nu} = -\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{\not p - \not k_1 - m}\gamma_\mu \underset{\uparrow}{-} \frac{i}{\not p - \not k_1 - m}\gamma_5\gamma_\mu \frac{i}{\not p - \not q - m}\gamma_\nu \right.$$
$$\left. + \frac{i}{\not p - m}\gamma_5\gamma_\mu \frac{i}{\not p - \not k_2 - m}\gamma_\nu \underset{\uparrow}{-} \frac{i}{\not p - \not k_2 - m}\gamma_5\gamma_\nu \frac{i}{\not p - \not q - m}\gamma_\mu \right]$$

If we can shift integration variable in the 1st term, $p \to p - k_2$ then it cancel against the forth term.

Similarly, 2nd & 3rd term cancel.

However, $\Delta_{\mu\nu}$ is linear divergent, $\left|\ \because\ \mathrm{tr}(\not p\,\gamma_5\gamma_\nu\not p\,\gamma_\mu) \propto \underbrace{\mathrm{tr}(\gamma_\alpha\gamma_5\gamma_\nu\gamma_\beta}_{\varepsilon_{\alpha\nu\beta\mu}}\gamma_\mu)\, p_\alpha p_\beta \right.$

$$= 0$$

$$\therefore\ \Delta_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}\left(g^{\mu\nu}(p - k_2) - g^{\mu\nu}(p) \right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{pmatrix}$$

where $g^{\mu\nu}(p) = \mathrm{tr}\left(\frac{1}{\not p - m}\gamma_5\gamma_\nu \frac{1}{\not p - \not k_1 - m}\gamma_\mu \right) \sim O(p^{-3})$ for $p \to \infty$

In general, $\int d^nx \left[f(x+a) - f(x) \right]$

$$= \int d^nx \left(a^\mu \partial_\mu f + a^\mu \partial_\mu a^\nu \partial_\nu f + \cdots \right)$$

$$= \lim_{R\to\infty} \int^R d^nx \left(a^\mu \partial_\mu f + \underbrace{a^\mu \partial_\mu a^\nu \partial f + \cdots}_{} \right)$$

$\hookrightarrow$ drop out $f \sim \frac{1}{R^{n-1}}$ for large R

$$= \lim_{R\to\infty} S_n(R)\, a^\mu \frac{R^\mu}{R} f(R)$$

$\Delta_{\mu\nu} \neq 0$ because of Surface contribution.

$$\Delta_{\mu\nu} = -k_2^{\sigma} \int \frac{d^4p}{(2\pi)^4} \frac{\partial}{\partial p^{\sigma}} \left(\frac{\mathrm{tr}\left[(\not{p}+m)\ \gamma_5 \gamma_\nu (\not{p}-\not{k}_1 - m)\right]\gamma_\mu}{(p^2-m^2)\left[(p-k_1)^2-m^2\right]} \right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \end{pmatrix}$$

$$= \frac{-k_2^{\sigma}}{(2\pi)^4} \lim_{p\to\infty} \frac{p_\sigma}{p}\ \underbrace{\mathrm{tr}\,(\gamma_\alpha \gamma_5 \gamma_\nu \gamma_\beta \gamma_\mu)}_{4i\varepsilon_{\nu\beta\mu\alpha}}\ \frac{p^{\alpha} k_1^{\beta}}{p^4} \cdot 2\pi^2 i p^3 + (\sim)$$

i due to Wick rot.

$$= -\frac{1}{8\pi^2} k_1^{\alpha} k_2^{\beta} \varepsilon_{\mu\nu\alpha\beta} + (\sim)$$

$$\lim_{p\to\infty} \frac{p^{\sigma}p^{\alpha}}{p^2} \to \frac{1}{4}\eta^{\sigma\alpha}$$

in the integral

$$= -\frac{1}{4\pi^2} k_1^{\alpha} k_2^{\beta} \varepsilon_{\mu\nu\alpha\beta}$$

$$\therefore\ q^{\lambda} T_{\mu\nu\lambda} = 2m T_{\mu\nu} \underbrace{- \frac{1}{4\pi^2} \varepsilon_{\mu\nu\alpha\beta}\ k_1^{\alpha} k_2^{\beta}}\ //$$

$\leftarrow$ Axial Ward id.

Similarly,

$$k_1^{\mu} T_{\mu\nu\lambda} = (-1) \int \frac{d^4p}{(2\pi)^4} \left\{ \begin{aligned} &\mathrm{tr}\left[\frac{1}{\not{p}-m} \gamma_\lambda \gamma_5 \frac{1}{\not{p}-\not{q}-m} \gamma_\nu \frac{1}{\not{p}-\not{k}_1-m} \not{k}_1\right] \\ &+ \mathrm{tr}\left[\frac{1}{\not{p}-m} \gamma_\lambda \gamma_5 \frac{1}{\not{p}-\not{q}-m} \not{k}_1 \frac{1}{\not{p}-\not{k}_2-m} \gamma_\nu\right] \end{aligned} \right\}$$

using $\not{k}_1 = (\not{p}-m) - (\not{p}-\not{k}_1-m) = (\not{p}-\not{k}_2-m) - (\not{p}-\not{q}-m)$

$$k_1^{\mu} T_{\mu\nu\lambda} = (-1) \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left[\gamma_\lambda \gamma_5 \frac{1}{\not{p}-\not{q}-m} \gamma_\nu \frac{1}{\not{p}-\not{k}_1-m} - \gamma_\lambda \gamma_5 \frac{1}{\not{p}-\not{k}_2-m} \gamma_\nu \frac{1}{\not{p}-m}\right]$$

$$= \frac{k_1^{\sigma}}{(2\pi)^4} \cdot 2i\pi^2 \lim \frac{p_\sigma}{p^2}\ \mathrm{tr}\left(\gamma_\lambda \gamma_5 \gamma_\alpha \gamma_\nu \gamma_\beta\right) k_2^{\alpha} p^{\beta}$$

$$= -\frac{1}{8\pi^2} \varepsilon_{\lambda\alpha\nu\beta} k_1^{\beta} k_2^{\alpha}$$

$$= \frac{1}{8\pi^2} \varepsilon_{\nu\lambda\alpha\beta} k_1^{\alpha} k_2^{\beta} //$$

Vector Ward id.

Pb? Both Ward identities are violated?!

Ambiguity in $T_{\mu\nu\lambda}$ (Routing of momentum)

Suppose the loop momentum is $p+a$, $a = \alpha k_1 + (\alpha-\beta) k_2$ where α, β arbitrary.

Define $\Delta_{\mu\nu\lambda}(a) \hat{=} T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)$

If $\Delta_{\mu\nu\lambda}(a) \neq 0$, then we see that $T_{\mu\nu\lambda}$ depends on how we label the $_\wedge^{\text{loop}}$ mom.

$$\underbrace{\Delta_{\mu\nu\lambda}(a) = (-1)\int \frac{d^4p}{(2\pi)^4} \left\{ \begin{array}{l} tr\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{\not p + \not a - \not q - m}\gamma_\nu \frac{1}{\not p + \not a - \not k_1 - m}\gamma_\mu\right] \\ - tr\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{\not p - \not q - m}\gamma_\nu\frac{1}{\not p - \not k_1 - m}\gamma_\mu\right] \end{array}\right\}}_{} + \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \to k_2 \end{pmatrix}$$

$$(-1)\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} tr\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{\not p - \not q - m}\gamma_\nu\frac{1}{\not p - \not k_1 - m}\gamma_\mu\right]$$

$$= -\frac{i\, 2\pi^2 a^\sigma}{(2\pi)^4} \lim_{p\to\infty} p^2 p_\sigma \underbrace{\frac{tr(\gamma_\alpha\gamma_\lambda\gamma_5 \not p\, \gamma_\nu \not p\, \gamma_\mu)\, p^\alpha}{p^6}}_{}$$

$$\not p \gamma_\nu \not p = 2\not p\, p_\nu - p^2\gamma_\nu$$

$$\lim_{p\to\infty} \frac{1}{p^4} 2p_\sigma\, tr(\gamma_\alpha\gamma_\lambda\gamma_5\gamma_\beta\gamma_\mu) \underset{\sim\sim}{p^\alpha p^\beta} p_\nu - \frac{1}{p^2} p_\sigma\, tr(\gamma_\alpha\gamma_\lambda\gamma_5\gamma_\nu\gamma_\mu)\, p^\alpha.$$

$$= 0 \quad - i\varepsilon_{\nu\mu\sigma\lambda}$$

$$= \frac{-1}{8\pi^2}\varepsilon_{\mu\nu\lambda\sigma}\, a^\sigma$$

$$\therefore \Delta_{\mu\nu\lambda}(a) = \frac{-1}{8\pi^2}\varepsilon_{\mu\nu\lambda\sigma}\left[\left(\alpha k_1^\sigma + (\alpha-\beta)k_2^\sigma\right) - \left(\alpha k_2^\sigma + (\alpha-\beta)k_1^\sigma\right)\right] = \frac{-\beta}{8\pi^2}\varepsilon_{\mu\nu\lambda\sigma}(k_1^\sigma - k_2^\sigma)$$

ie. $\underline{T_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(0) - \frac{\beta}{8\pi^2}\varepsilon_{\mu\nu\lambda\sigma}(k_1-k_2)^\sigma}$

$$\Rightarrow \begin{cases} q^\lambda T_{\mu\nu\lambda}(\beta) = 2m T_{\mu\nu}(0) - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \\ k_1^\mu T_{\mu\nu\lambda}(\beta) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\alpha\beta}k_1^\alpha k_2^\beta \end{cases}$$

We can choose $\beta = -1$ such that VW is preserved.

But then AW has an anomaly.

$\nwarrow$ important as J_μ generate U(1) of QED

Gauge symm. must not be broken.

Otherwise quantizing gauge theory will

Anomalous divergence of Axial current

For the choice $\beta = -1$,

$$\begin{cases} q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{1}{2\pi^2} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \\ k_1^\mu T_{\mu\nu\lambda} = 0 \end{cases}$$

The first equation means that $\partial_\mu J_5^\mu$ can couple to two J^μ by a derivative coupling. Since J^μ couple to A^μ with coupling $(-ie)$, we conclued that

$\partial_\mu J_5^\mu$ can create 2 photons (lowest order)

let $\varepsilon_\mu(k_1)$, $\varepsilon_\nu(k_2)$ be the polarization of the two photons, then

$$\int d^4x\, e^{-iq\cdot x} \langle k_1, k_2 | J_5^\lambda(x) | 0 \rangle = (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q)\, \varepsilon_\mu^*(k_1)\, \varepsilon_\nu^*(k_2) \left(-i T_{\mu\nu\lambda}(k_1, k_2, q) \right) (-ie)^2$$

$+iq_\lambda \int$

$$\int d^4x\, e^{-iq\cdot x} \langle k_1, k_2 | \partial_\lambda J_5^\lambda | 0 \rangle = \text{"}m\text{"} + (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q) - \frac{1}{2\pi^2} \varepsilon_{\mu\nu\alpha\beta}\, k_1^\alpha \varepsilon^{\mu *}(k_1)\, k_2^\beta \varepsilon^{\nu *}(k_2)\, (-ie)^2$$

$$= \text{"}m\text{"} + (2\pi)^4 \delta^{(4)}(k_1 + k_2 - q) \times \frac{-e^2}{2\pi^2} \varepsilon_{\mu\nu\alpha\beta} (-ik_1^\alpha) \varepsilon^{\mu *}(k_1) (-ik_2^\beta) \varepsilon^{\nu *}(k_2)$$

$$\Rightarrow \langle k_1, k_2 | \partial_\lambda J_5^\lambda(0) | 0 \rangle = \text{"}m\text{"} + \frac{e^2}{16\pi^2} \varepsilon^{\alpha\mu\beta\nu} \langle k_1, k_2 | F_{\alpha\mu} F_{\beta\nu}(0) | 0 \rangle$$

ie $$\boxed{\partial_\lambda J_5^\lambda = 2im P + \frac{e^2}{16\pi^2} \varepsilon^{\alpha\mu\beta\nu} F_{\alpha\mu} F_{\beta\nu}}$$

rmk : 1° The anomaly (second term) is indep of fermion mass and metric

Topological origin ?

2° Adler and Bardeen showed that this 1-loop result is exact.

No higher loop correction

Fujikawa method

- Consider the functional integral

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; e^{i\int \bar{\psi}\, i\not{D}\, \psi} \qquad\qquad \not{D} = \gamma^\mu(\partial_\mu + A_\mu)$$

The measure is defined by :

Let ϕ_m be an eigenvectr of $i\not{D}$, $i\not{D}\phi_m = \lambda_m \phi_m$

then one can expand ψ, $\bar{\psi}$ as

$$\psi = \sum a_i \phi_i$$
$$\bar{\psi} = \sum \bar{b}_i \phi_i^\dagger$$

where $a_i, \bar{b}_i$ are anticommuting Grassman variables

$$\{a_i, a_j\} = \{\bar{b}_i, \bar{b}_j\} = \{a_i, \bar{b}_j\} = 0$$

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \triangleq \prod_i da_i \, d\bar{b}_i$$

Now, under a chiral rotation,

$$\psi \to \psi' = (1 + i\alpha(x)\gamma^5)\,\psi(x)$$
$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\,(1 + i\alpha(x)\gamma^5)$$

$$\int \bar{\psi}' i\not{D} \psi' = \int \bar{\psi} i\not{D}\psi - \partial_\mu \alpha\, \bar{\psi}\gamma^\mu\gamma^5\psi$$
$$= \int \bar{\psi}\, i\not{D}\psi + \alpha\, \partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi)$$
$$= \int \bar{\psi}\, i\not{D}\psi + \int \alpha\, \partial_\mu j^\mu_5$$

To find out how the measure change, we note that

$$\psi' = \sum a_i' \phi_i \quad \Rightarrow \quad a_i' = \langle \phi_i | \psi' \rangle = \langle \phi_i | (1 + i\alpha\gamma^5) | \psi \rangle$$
$$= \underbrace{\langle \phi_i | 1 + i\alpha\gamma^5 | \phi_j \rangle}_{C_{ij}} a_j$$

$$\therefore \; \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \underset{\substack{\uparrow \\ \text{Grassmanian}}}{\det(1+C)^{-2}} \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

$$\det(1+C)^{-1} = \exp(-\mathrm{tr}\ln C\ )$$

$$= \exp\left[-\mathrm{tr}\ln\left(1+i\alpha\langle\phi_i|\gamma^5|\phi_j\rangle\right)\right]$$ keeping only linear term in α

$$= \exp\left(-i\alpha\sum_i\langle\phi_i|\gamma^5\phi_i\rangle\right)$$

$$\therefore\ \mathcal{D}\psi'\mathcal{D}\bar\psi' = \mathcal{D}\psi\mathcal{D}\bar\psi\ e^{-2i\alpha A(x)} \qquad A(x) = \sum_i \phi_i^\dagger\gamma^5\phi_i(x)$$

We can think of the chiral rotation as a change of variables in Z, since α is arbitrary, so

$$\partial_\mu j^5_\mu = -2i\,A(x)$$

To calculate A(x),

Formally, $$A(x) = \sum_n \phi_n^\dagger(x)^{\dot\alpha}\,(\gamma_5)_{\dot\alpha}{}^{\alpha}\,\phi_n(x)_\alpha$$

$$= \underbrace{\sum_n \phi_n^\dagger(x)^{\dot\alpha}\,\phi_n(x)_\alpha}_{\delta^{\dot\alpha}_\alpha\,\delta(0)}\,(\gamma_5)_{\dot\alpha}{}^{\alpha}$$

$$= \delta(0)\ \mathrm{tr}\,\gamma_5$$

$$= \infty \times 0$$

We must regulate it to get a sensible result.

We use the Heat Kernel regularization

$$\sum_n \phi_n^\dagger\gamma_5\phi_n = \lim_{M\to\infty}\phi_n^\dagger(x)\gamma_5\,e^{-\lambda_n/M^2}\,\phi_n(x)$$

$$= \cdots$$ (see next page)

$$= -\frac{1}{32\pi^2}\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,F_{\mu\nu}F_{\alpha\beta}$$

$$\therefore\ \partial_\mu j^\mu_5 = \frac{1}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,F_{\mu\nu}F_{\alpha\beta}$$

//

For U(1), $\mathrm{tr}\,T^0T^0 = 1$ reproduce previous result.

For G, $\mathrm{tr}\,T^aT^b = \frac{1}{2}\delta^{ab}$

Evaluating $\sum_n \phi_n^+ \gamma_5 \phi_n$

$$\sum_n \phi_n^+ \gamma_5 \phi_n = \lim_{M\to\infty} \sum_n \phi_n^+ \gamma_5 \, e^{-\lambda_n^2/M^2} \phi_n$$

$$= \quad '' \quad e^{-\not{D}^2/M^2} \quad ''$$

$$= \lim_{M\to\infty} \sum_n \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\varphi}_n^+(k) \gamma_5 \, e^{-\not{D}^2/M^2} \int \frac{d^4k'}{(2\pi)^4} e^{ik'x} \tilde{\varphi}_n(k')$$

$$= \lim_{M\to\infty} \sum_n \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \Big(\underbrace{\sum_n \hat{\varphi}_n^+(k)_\alpha \, \hat{\varphi}_n^+(k')^{\dot\alpha}}_{(2\pi)^4 \, \delta^4(k-k') \, \delta_\alpha^{\dot\alpha}} \Big) e^{-ikx} \left(\gamma_5 \, e^{-\not{D}^2/M^2} \right)^\alpha_{\dot\alpha} e^{ik'x}$$

$$= \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \, \mathrm{tr}\left(\gamma_5 \, e^{-\not{D}^2/M^2} \right) e^{ikx}$$

$$= \lim_{M\to\infty} \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\Big(\gamma_5 \, e^{\underbrace{-(\not{D}+i\not{k})^2/M^2}} \Big)$$

$$\Vert$$

$$\frac{-\not{D}^2 - i\{\not{D}, \not{k}\} + \not{k}^2}{M^2}$$

$$= \frac{-D^2 - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - 2i\,k\cdot D}{M^2} - k^2/M^2$$

$$= \lim_{M\to\infty} M^4 \int \frac{d^4k}{(2\pi)^4} e^{-k^2} \, \underbrace{\mathrm{tr}\left[\gamma_5 \exp\left(-\frac{2ik\cdot D}{M} - \frac{D^2}{M^2} - \frac{\gamma^\mu\gamma^\nu F_{\mu\nu}}{2M^2} \right) \right]}_{\mathrm{tr}\, \gamma_5 \left(\cdots + \frac{1}{2!} \frac{(\gamma_\mu \gamma_\nu F_{\mu\nu})^2}{4M^4} + \cdots \right)}$$

$$= \frac{1}{8} \underbrace{\left(\int \frac{d^4k}{(2\pi)^4} e^{-k^2} \right)}_{\frac{1}{(2\pi)^4}\left(\int_{-\infty}^{\infty} dq \, e^{-q^2} \right)^4} \underbrace{\mathrm{tr}\left(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \right)}_{4i\,\varepsilon^{\mu\nu\alpha\beta}} \mathrm{tr}\left(F_{\mu\nu} F_{\alpha\beta} \right)$$

$$=$$

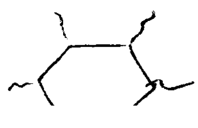
$$= -\frac{1}{32\pi^2} \varepsilon^{\mu\nu\alpha\beta} \, \mathrm{tr}\left(F_{\mu\nu} F_{\alpha\beta} \right) \; //$$

The coupling e^2 can be put back if we remember that

$$D_\mu = \partial_\mu + e A_\mu$$

this method can be easily generalized to higher dimensions

$$\partial_\mu j^\mu_{n+1} = \frac{2e^n}{2^n} \cdot \frac{1}{n!} \, \mathrm{tr}\left(\frac{iF}{2\pi} \right)^n \quad , \quad D = 2n$$

perturbatively, corresponds to polygonal diagrams.  in 10 dim.

$\boxed{\pi^0 \to 2\gamma \text{ decay and PCAC}}$

	Q
u	$2/3$
d	$-1/3$

$\pi^{\pm} = u\bar{d},\ \bar{u}d$

$\pi^0 = \bar{u}u + \bar{d}d$

let π^i $i=1,2,3$ denote the pion state

$$J^k_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\sigma^k\psi$$

$J^k_{5\mu}$ can annihilate a pion state,

$$\langle 0|J^k_{5\mu}|\pi^j\rangle = i\,\underbrace{\delta_{jk}}_{\substack{\uparrow\\ \text{isospin}\\ \text{conservation}}} f_\pi \underbrace{p_\mu}_{\substack{\uparrow\\ \text{Lorentz invariance}}} e^{-ip\cdot x}$$

$$\langle 0|\partial^\mu J^k_{5\mu}|\pi^j\rangle = \delta_{jk} f_\pi m_\pi^2 e^{-ip\cdot x}$$

$\Rightarrow$ we can define $\boxed{\phi^k_\pi(x) = \frac{1}{m_\pi^2 f_\pi}\partial^\mu J^k_{5\mu}(x)}$ as the pion operator

$(\because \langle 0|\phi^k_\pi(x)|\pi^j\rangle = \delta_{jk}\, e^{-ip\cdot x}$

rmk: 1° It is called PCAC relation (Partially conserved Axial current)

2° It reflects the bound state nature of pion.

Consider the reaction $a \to b + \pi^k$

reduction formula:

$$\langle \pi^k, b^{out}|a^{in}\rangle = i\int d^4x\, e^{i\underset{\uparrow}{p}\cdot x}(\Box + m_\pi^2)\langle b|\phi^k_\pi(x)|a\rangle$$

p = momentum of pion

$$\text{Amp}(a\to b+\pi^k) = (m_\pi^2 - p^2)\langle b|\phi_\pi^k(0)|a\rangle$$

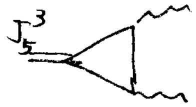
Using PCAC,

$$\text{Amp}(a\to b+\pi^k) = \frac{m_\pi^2 - p^2}{p^2 f_\pi}\langle b|\partial^\mu J^k_{5\mu}(0)|a\rangle$$

$$\simeq -\frac{1}{f_\pi}\langle b|\partial^\mu J^k_{5\mu}(0)|a\rangle \qquad m_\pi^2 \simeq 0$$

$f_\pi \sim 93\,\text{MeV}$

$\pi^0 \to 2\gamma$



$$J^3_{5\mu} = \bar u \gamma_\mu \gamma_5 u - \bar d \gamma_\mu \gamma_5 d$$

$$\partial^\mu J^3_{5\mu} = 2m_u(\bar u \gamma_5 u) - 2m_d(\bar d \gamma_5 d) + \frac{e^2}{16\pi^2}\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,\mathrm{tr}(\sigma^3 Q^2)$$

$\nwarrow \; Q = \begin{pmatrix} 2/3 & \\ & -1/3 \end{pmatrix}$

$$\mathrm{tr}(\sigma^3 Q^2) = \underset{\uparrow}{3} \cdot \left[(2/3)^2 - (1/3)^2\right] = 1$$

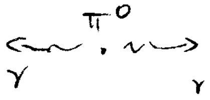
colors of quarks

$$\therefore \partial^\mu J^3_{5\mu} = \frac{\alpha}{4\pi}\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} + m\text{'s}$$

Decay amplitude $= A = -\frac{1}{f_\pi}\langle k_1\varepsilon_1; k_2\varepsilon_2 \,|\, J^3_{5\mu} \,|0\rangle$

$$= -\frac{2\alpha}{\pi f_\pi}\varepsilon_{\mu\nu\alpha\beta}k_1^\mu k_2^\nu \varepsilon_1^\alpha \varepsilon_2^\beta$$

If we introduce quantities in C-M frame,



$k_1 = (\omega, \vec k)$, $k_2 = (\omega, -\vec k)$, $|\vec k| = \omega$

$\varepsilon_1 = (0, \vec\varepsilon_1)$, $\varepsilon_2 = (0, \vec\varepsilon_2)$

then $\varepsilon_{\mu\nu\alpha\beta}k_1^\mu k_2^\nu \varepsilon_1^\alpha \varepsilon_2^\beta = 2\omega^2|\vec\varepsilon_1 \times \vec\varepsilon_2|$

$\vec k = k_x \hat e_x = \omega \hat e_x$

$\vec k \cdot \vec\varepsilon = 0$

$$\Rightarrow A = -\frac{4\alpha}{\pi f_\pi}\omega^2|\vec\varepsilon_1 \times \vec\varepsilon_2|$$

$$\Rightarrow \text{Rate}(\pi^0 \to 2\gamma) = \frac{\alpha^2}{64\pi^3}\left(\frac{m_\pi}{f_\pi}\right)^2 m_\pi$$

//

π^0 Life time $\tau = (8.5 \text{ eV})^{-1}$

experimental value $\tau_{expt} = [(7.95 \pm 0.55)\text{ eV}]^{-1}$

$= (0.828 \pm 0.057) \times 10^{-16}$ s //

Chiral Anomaly

Nonabelian Chiral Anomaly

- So far we have coupled gauge field to fermions in a Parity symmetric manner. ie ψ_L & ψ_R has the same coupling.

However, this only gives a subset of the possible couplings of fermions to gauge fields:

$$\text{Dirac fermion } \psi = \psi_L + \psi_R \qquad \psi_L = \left(\frac{1-\gamma_5}{2}\right)\psi = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix}$$

↗ ↗

Weyl fermions

$$\psi_R = \left(\frac{1+\gamma_5}{2}\right)\psi = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix}$$

$$\gamma_5 \psi_R = +\psi_R$$
$$\gamma_5 \psi_L = -\psi_L$$

ψ_R & ψ_L can couple to gauge field diffently.

We will be interested in the anomaly from the loop of chiral fermion.

Chiral fermions exists in standard model, e.g. $e_L = \underline{2}$ of $SU(2)$
$e_R = \underline{1}$ of $SU(2)$.

⇒ Important to be sure that standard model is anomaly free!

- Since Left handed & Right handed fermion coupled independently to A_μ, we can restrict our attention to the Lagrangian

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) P_+ \psi \qquad P_\pm = \frac{1 \pm \gamma_5}{2}$$

where $A_\mu = A_\mu^a T^a$ and

ψ^i is in reps. $\underline{v}$ of the Lie alg. $\mathfrak{g}$, and $\gamma^5 \psi = \psi$

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu^a T^a)\left(\frac{1+\gamma^5}{2}\right) \psi$$

Classically, $\mathcal{L}$ is inv. under

$$\begin{cases} \delta\psi = -v\psi \\ \delta A_\mu = \partial_\mu v + [A_\mu, v] \end{cases} \qquad v = v^a T^a$$

- We have shown that for Dirac fermion ψ,

$$e^{-\Gamma[A]} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; e^{-\int \bar{\psi}\, i\not{D}\, \psi} = \det(i\not{D})$$

Variation of the measure gives

$$\partial_\mu j^\mu_5 = -2 \sum_n \phi_n^\dagger(x)\, \gamma^5\, \phi_n(x)$$

where $i\not{D}\,\phi_n = \lambda_n \phi_n$ $i\not{D}$ Hermitian
λ_n real.

- Now we have Weyl fermions, the eigenvalue problem:

$$\underset{\uparrow}{\underbrace{i\not{D}_+}}\,\psi = \lambda \underset{\uparrow\ \gamma^5\psi = \psi}{\psi} \quad \text{is not well defined}$$

$$\gamma^5(\not{D}_+\psi) = -\not{D}_+(\gamma_5\psi) = -\not{D}_+\psi . \qquad \text{negative chirality} .$$

We can remedy this situation by defining $\Gamma[A]$ by

$$e^{-\Gamma[A]} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; e^{-\int \bar{\psi}\, i\hat{\mathcal{D}}\, \psi}$$

where $\hat{\mathcal{D}} = i\gamma^\mu(\partial_\mu + A_\mu P_+) = i\gamma^\mu(\partial_\mu + A_\mu)P_+ + i\gamma^\mu\partial_\mu P_-$

$$\bar{\psi}\, i\hat{\mathcal{D}}\,\psi = \underbrace{\bar{\psi}\, i\gamma^\mu(\partial_\mu + A_\mu)P_+\psi}_{\overline{(\psi_+)}\, i\not{D}\, \psi_+} + \underbrace{\bar{\psi}\, i\gamma^\mu\partial_\mu P_-\psi}_{\underset{\uparrow\ \gamma^5 = 1}{\underbrace{\overline{(\psi_-)}}}\, i\not{\partial}\, \underset{\uparrow\ \gamma^5 = 1}{\underbrace{\psi_-}}} \qquad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Since $\overline{(\psi_-)} = (\psi_-)^\dagger \gamma_{\cdot} = \psi^\dagger(\frac{1-\gamma_5}{2})\gamma_0$
$= \bar{\psi}\,\frac{1+\gamma_5}{2} = (\bar{\psi})_+$

The second piece contribute a trivial A-indep factor to $e^{-\Gamma}$!

$$\Rightarrow\ e^{-\Gamma[A]} = \det(i\hat{\mathcal{D}})$$

Now $i\hat{\mathcal{D}} = \begin{array}{c|cc} & + & - \\ \hline + & 0 & i\not\partial_- \\ - & i\not D_+ & 0 \end{array}$ has a well-defined eigen value problem.

$$\begin{cases} i\hat{\mathcal{D}}\,\phi_n = \lambda_n \phi_n \\ \chi_n^+ (i\overleftarrow{\hat{\mathcal{D}}}) = \lambda_n \chi_n^+ \qquad \Leftrightarrow \quad (i\hat{\mathcal{D}})^+ \chi_n = \bar{\lambda}_n \chi_n \end{cases}$$

$\lambda_n \neq \bar{\lambda}_n$ since $i\hat{\mathcal{D}}$ is not Hermitian.

Can choose $\int \chi_n \phi_m^+ = \delta_{nm}$

- Is λ_n or $\det(i\hat{\mathcal{D}})$ gauge inv. ?

No, since $i\hat{\mathcal{D}}(A^g) \neq g\, i\hat{\mathcal{D}}(A)\, g^{-1}$ $\qquad$ $i\not D_+(A^g) = g\, i\not D_+(A)\, g^{-1}$

therefore $\Gamma[A]$ is not gauge inv. $\qquad$ but $i\not\partial_- \neq g\, i\not\partial_-\, g^{-1}$

$\Rightarrow$ Anomaly!

$$\text{Now } \det(i\hat{\mathcal{D}})\det(i\hat{\mathcal{D}})^+ = \det\left[i\hat{\mathcal{D}}\,(i\hat{\mathcal{D}})^+\right] = \det\begin{pmatrix} i\not\partial_-\, i\partial_+ & 0 \\ 0 & i\not D_+\, i\not D_- \end{pmatrix} = \underbrace{\det(i\not\partial_-\, i\not\partial_+)}_{\text{gauge inv.}}\underbrace{\det(i\not D_+}_{\text{claim: also gauge inv.}}$$

Why? since the Dirac op. $i\not D = \begin{pmatrix} 0 & i\not D_- \\ i\not D_+ & 0 \end{pmatrix}$

has gauge inv. det $\quad (\det(i\not D))^2 = \det\begin{pmatrix} i\not D_-\, i\not D_+ & \\ & i\not D_+\, i\not D_- \end{pmatrix} = \left[\det(i\not D_+\, i\not D_-)\right]^2$

$\therefore\ |\det(i\hat{\mathcal{D}})|$ is gauge inv.

ie $\mathrm{Re}\,\Gamma(A)$ is gauge inv.

$$\boxed{\text{only } \mathrm{Im}\,\Gamma[A] \text{ may suffer from an anomaly.}}$$

- Following previous calculation, we have

under $\begin{cases} A \to A + Dv \\ \psi \to \psi - v\psi \\ \bar{\psi} \to \bar{\psi} + \bar{\psi} v \end{cases}$

The Jacobian factor transform by

$$\prod da_i \, d\bar{b}_i \to \prod da_i \, d\bar{b}_i \; e^{-2i\int \mathrm{tr}\, v \underbrace{\sum_n \chi_n^+ \gamma^5 \phi_n(x)}_{\text{Natively } \delta(0) \times \mathrm{tr}\,\gamma^5}}$$

Gross + Jackiw (1972) evaluate this using the regulator $e^{-(i\hat{D})^2/M^2}$

and find

$$\boxed{\begin{aligned} \Gamma[A+Dv] - \Gamma[A] &= \int dx \, \mathrm{tr}(v \, D_\mu j^\mu) \qquad\qquad j^{\mu a} = \bar{\psi}_+ \gamma^\mu T^a \psi_+ \\ &= \frac{1}{24\pi^2} \int dx \, \mathrm{tr}\left[v \, \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \left(A_\beta \partial_\gamma A_\delta + \frac{1}{2} A_\beta A_\gamma A_\delta \right) \right] \\ (D_\mu j^\mu)^a &= \frac{1}{24\pi^2} \mathrm{tr}\left[T^a \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \left(A_\beta \partial_\gamma A_\delta + \frac{1}{2} A_\beta A_\gamma A_\delta \right) \right] \end{aligned}}$$

rmk

1° In terms of Feynman diagrams, triangular as well as box diagrams both contributes.

2° If we consider ψ_- instead, we get a negative sign.

i.e. $(D_\mu j^\mu_+)^a = \frac{1}{24\pi^2} \mathrm{tr}[\cdots]$

$(D_\mu j^\mu_-)^a = -\frac{1}{24\pi^2} \mathrm{tr}[\cdots]$