AM2. Final exam.

1/2015

1. $\underbrace{y' + 2xy}$ $= xe^{-x^2}$

interprating factor [5]

$e^{-x^2}(e^{x^2}y)'$ $\Rightarrow$ $(e^{x^2}y)' = x$

$e^{x^2}y = \frac{1}{2}x^2 + c$

c = constant.

$y = e^{-x^2}\left(\frac{1}{2}x^2 + c\right)$ // [5]

2. $y'' + y' - 2y = 18xe^{x}$

C.F., Put $y = e^{\lambda x}$, $\Rightarrow$ $\lambda^2 + \lambda - 2 = 0$

$\lambda = 1$ or -2

$y_{CF} = c_1 e^{x} + c_2 e^{-2x}$ // [5]

P.I., Try $y_{PI} = xe^{x}(ax+b)$ [2]

$y'_{PI} = e^{x}(ax^2 + bx + 2ax + b)$

$y''_{PI} = e^{x}(ax^2 + bx + 4ax + 2b + 2a)$

$\therefore y''_{PI} + y'_{PI} - 2y_{PI} = e^{x}(6ax + 3b + 2a) \doteq 18xe^{x}$

$\Rightarrow \begin{cases} 6a = 18 \\ 3b + 2a = 0 \end{cases}$

$\Rightarrow$ $a = 3$

$b = -2$ // [3]

3.a. $\nabla^2 u = 0$

$\therefore u$ is the real part of a analytic function [2]

C.R. condition

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

1st eqn $\Rightarrow$ $v_y = e^{-y}\cos x$

$\Rightarrow$ $v = -e^{-y}\cos x + g(x)$

2nd eqn $\Rightarrow$ $-e^{-y}\sin x = -\left(e^{-y}\sin x + g'(x)\right)$

$\Rightarrow g'(x) = 0 \quad \Rightarrow g = \text{Const.}$

$\therefore v = -e^{-y}\cos x + C.$

$$f = e^{-y}\underbrace{(\sin x - i\cos x)}_{-ie^{ix}} + C$$

$$= -ie^{i(x+iy)} + C$$

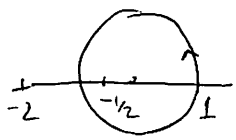
$$= -ie^{iz} + C$$ [3]

b. $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

For $z = e^{i\theta}$, $\quad \dfrac{dz}{z} = i\,d\theta$

$\cos\theta = \dfrac{z+z^{-1}}{2}$

$$I = \frac{1}{i}\oint_C \frac{dz}{z}\frac{1}{5+2(z+\frac{1}{z})} = \frac{1}{i}\oint dz \cdot \frac{1}{2z^2+5z+2}$$



root of $2z^2+5z+2=0$ is $z = \dfrac{-5 \pm \sqrt{25-16}}{4} = -\frac{1}{2}$ or -2 [3]

$$\text{Res}\left(\frac{1}{(2z^2+5z+2)}, z=-\tfrac{1}{2}\right) = \text{Res}\left(\frac{1}{2(z-\alpha)(z-\beta)}, z=\alpha\right) = \frac{1}{2}\frac{1}{\alpha-\beta} = \frac{1}{2}\left(\frac{1}{-\frac{1}{2}+2}\right) = \frac{1}{3}$$

$I = 2\pi/3$ [2]

4.a. $z=0$ pole of order 1

$z=1$ "

[4]

b. $f = \frac{1}{z(1-z)} = -\frac{1}{z-1} \cdot \frac{1}{1+(z-1)}$

$= -\frac{1}{z-1} \sum_{n=0}^{\infty} (z-1)^n (-1)^n$ for $|z-1| < 1$

$= -\frac{1}{z-1} + 1 - (z-1) + (z-1)^2 + \cdots$ //

[6]

5. a. $y'' - \overset{p}{\frac{3z}{1-z^2}} y' + \overset{q}{\frac{\lambda}{1-z^2}} y = 0$

$z=0$ is a ordinary point since p, q are finite at $z=0$. [3]

$y = \sum_{n=0}^{\infty} a_n z^n$

$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}$

$y' = \sum_{n=0} n a_n z^{n-1}$

$$\Rightarrow (1-z^2)y'' - 3z y' + \lambda y = \underbrace{\sum_{n=0} n(n-1) a_n z^{n-2}}_{\substack{\sum_{n=2} a_n z^{n-2} n(n-1) \\ = \sum_{n=0} a_{n+2}(n+2)(n+1)}} - \sum_{n=0} n(n-1) a_n z^n - 3\sum_{n=0} n a_n z^n + \lambda \sum a_n z^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - (n^2+2n-\lambda) a_n\right] z^n = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} = (n^2+2n-\lambda) a_n$$

$$a_{n+2} = \frac{n(n+2)-\lambda}{(n+2)(n+1)} a_n$$

[7]

The two power series are given by putting $a_0 = 1,\ a_1 = 0$

and $a_1 = 1,\ a_0 = 0$. //

b. ~~The~~ power series soln is obtained if the recursion relation truncated.

ie $\lambda = N(N+2)$ $\qquad N$ = integer.

In this case, $a_{N+2} = 0 \cdot a_N$

we obtain a soln of the form $y = \sum_{n=0}^{N} a_n z^n$ [5]

ie a polynomial of order N. //

6 a. $\frac{\partial \Phi}{\partial x} = (1-2xh+h^2)^{3/2} \cdot 3h = \sum_{n=0} h^n P_n'(x) = \sum_{n=1} h^n P_n'(x) \qquad \because P_0(x) = 1$

$\frac{\partial \Phi}{\partial h} = (1-2xh+h^2)^{3/2}\, 3(x-h) = \sum_{n=1} n h^{n-1} P_n(x)$

1st eqn $\Rightarrow 3(1-2xh+h^2)^{3/2} = \sum_{n=1}^{\infty} h^{n-1} P_n'(x)$

sub into 2nd eqn $\sum_{n=1} n h^{n-1} P_n(x) = (x-h)\sum_{n=1}^{\infty} h^{n-1} P_n'(x)$

$= \sum_{n=1} h^{n-1} x P_n' - \underbrace{\sum_{n=1} h^n P_n'(x)}_{\sum_{n=1} h^{n-1} P_{n-1}'(x)}$

$\Rightarrow xP_n' - P_{n-1}' = nP_n$ [5]

b. $n\int_{-1}^{1} P_n^2\, dx = \int_{-1}^{1} x \underbrace{P_n' P_n}_{\frac{1}{2}(P_n^2)'} dx - \int_{-1}^{1} P_{n-1}' P_n\, dx$

$= \frac{1}{2}\left\{ x(P_n^2)' \Big|_{-1}^{1} - \int_{-1}^{1} P_n^2\, dx \right\} - \int_{-1}^{1} P_{n-1}' P_n\, dx$ [2]

Put $x=1$ into def. of Φ, $\frac{1}{1-h} = \sum h^n P_n(1) \Rightarrow P_n(1) = 1$

Put $x=-1$, $\frac{1}{1+h} = \sum h^n P_n(-1) \Rightarrow P_n(-1) = (-1)^n$ [5]

$x(P_n^2)' \Big|_{-1}^{1} = 2$

$\Rightarrow (2n+1)\int_{-1}^{1} P_n^2 dx = 2 - 2\int_{-1}^{1} P_{n-1}' P_n \, dx$

Now P_{n-1}' is a polynomial of degree $n-1$

and it can be expanded as a sum of Legendre

Polynomial of degree $\leq n-1$

Using the orthogonality $\int_{-1}^{1} P_n P_m dx = 0$, [3]

Last term $= 0$

Hence the result! //

7. a. Separation of variables

Consider soln of the form $X(x)\,Y(y)$

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = c \quad \text{constant}$$

$$\Rightarrow \begin{cases} X'' + cX = 0 \\ Y'' - cY = 0 \end{cases}$$

$$c = 0, \quad \begin{cases} X = \alpha x + \beta \\ Y = \gamma y + \delta \end{cases} \qquad —① \qquad [2]$$

$$c = k^2 > 0, \quad \begin{cases} X = A \sin kx + B \cos kx \\ Y = c_1 e^{ky} + c_2 e^{-ky} = D_1 \sinh ky + D_2 \cosh ky \end{cases} \qquad —② \qquad [2]$$

$$c = -k^2 < 0, \quad \begin{cases} X = b_1 e^{kx} + b_2 e^{-kx} \\ Y = c \sinh ky + d \cosh ky \end{cases} \qquad —③ \qquad [2]$$

B.C. $X(0)\,Y(y) = 0 = X(a)\,Y(y) = 0 \;\Rightarrow\; X(0) = X(a) = 0$ —④

For soln ①, ④ implies that $\alpha = \beta = 0 \;\Rightarrow$ trivial soln [2]

Soln ③, ④ implies $b_1 = b_2 = 0 \;\Rightarrow$ "

only ② is possible, $B = 0$

$A \sin ka = 0 \Rightarrow \quad k = \frac{n\pi}{a} \qquad n = 0, \pm 1, \ldots$ [2]

Next B.C. at $y = 0$, $X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0$

sub. ②, we get $D_2 = 0$

$$\therefore \; V(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \quad //$$

indep. values of n.

$$\int_0^a \sin\frac{m\pi x}{a} V(x,y)\,dx = A_m \cdot \frac{a}{2} \cdot \sinh\frac{m\pi b}{a}$$

|| Put $y=b$

$$\int_0^a \sin\frac{m\pi x}{a}\Phi(x)\,dx \Rightarrow A_n = \frac{2}{a\sinh\frac{n\pi b}{a}} \cdot \int_0^a \sin\frac{n\pi x}{a}\Phi(x)\,dx$$

//

Put $\Phi = \begin{cases} A/2 & 0<x<a/2 \\ -A/2 & a/2\le x<a \end{cases}$

$$A_n = \frac{A}{a\sinh\frac{n\pi b}{a}}\Bigg(\underbrace{\int_0^{a/2}\sin\frac{n\pi x}{a} - \int_{a/2}^a \sin\frac{n\pi x}{a}\,dx}_{//}\Bigg)$$

$$\frac{a}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) - \frac{a}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$$= \frac{a}{n\pi}\underbrace{\left(1 - 2\cos\frac{n\pi}{2} + \cos n\pi\right)}_{= \begin{cases} 4 & n=4k+2,\ k=\text{integers} \\ 0 & \text{otherwise}\end{cases}}$$

$$\Rightarrow V = \frac{4A}{\pi}\sum_{k=0}^{\infty}\frac{1}{4k+2}\frac{1}{\sinh\frac{(4k+2)\pi b}{a}}\sin\frac{(4k+2)\pi x}{a}\sinh\left(\frac{4k+2}{a}\pi y\right)$$ [3]

//

b). $x = a/2$, $V=0$

$\Rightarrow V(\frac{a}{2}, b) = 0$ [2]

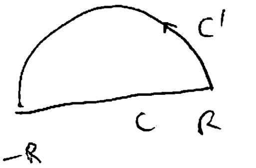
This is expected since this is what a F.S. converges to for a piecewise cont. function.

8\. a. Let U be a simply connected region of $\mathbb{C}$, f an analytic function except for poles $a_1, \dots, a_K$. If γ is a closed curve in U that enclosed the points $a_1, \dots, a_K$,

then $$\oint_\gamma f \, dz = 2\pi i \sum_{k=1}^{K} \operatorname{Res}(f, a_k)$$ [5]

b. $$I = \int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx}{a^2+x^2}$$

Consider $f = \dfrac{e^{imz}}{a^2+z^2}$ poles at $z = \pm ai$



$$\oint_{C+C'} = 2\pi i \operatorname{Res} = 2\pi i \cdot \frac{e^{im(ai)}}{2ai} = \frac{\pi}{a} e^{-ma}$$

$$\left|\int_{C'} f\right| \le \frac{1}{R} \to 0 \quad \text{as} \quad R \to \infty$$

$$\therefore \int_{-\infty}^{\infty} = \frac{\pi}{a} e^{-ma}$$

$$I = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} = \frac{\pi}{2a} e^{-ma}$$ [5]

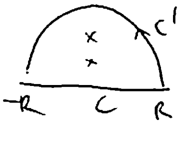
$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$$

Consider $f = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ pole at $z = \pm ai$, $z = \pm bi$

$$\operatorname{Res}(f, ai) = \frac{(ai)^2}{2ai(b^2-a^2)},$$

$$\operatorname{Res}(f, bi) = \frac{(bi)^2}{2bi(a^2-b^2)};$$

$$\Rightarrow 2\pi i\left[\operatorname{Res}(f, ai) + \operatorname{Res}(f, bi)\right] = \frac{\pi}{b^2-a^2}(b-a) = \frac{\pi}{b+a}$$



$$\left|\int_{C'}\right| \le \frac{1}{R} \to 0, \quad \Rightarrow I = \frac{\pi}{b+a}$$ [5]