1. (a) $y' + \frac{6x}{1+x^2} y = \frac{2x}{1+x^2}$

$I \triangleq \int \frac{6x}{1+x^2}\, dx = 3\ln(1+x^2)$

[5]

$e^I = (1+x^2)^3$

$(ye^I)' = y'e^I + I' y e^I = e^I(y' + \frac{6x}{1+x^2} y)$

$\Rightarrow (ye^I)' = e^I \cdot \frac{2x}{1+x^2} = (1+x^2)^2 \cdot 2x$

$\Rightarrow ye^I = \int 2x(1+x^2)^2\, dx = \frac{1}{3}(1+x^2)^3 + c$

[5]

$\Rightarrow y = \frac{1}{3} + \frac{c}{(1+x^2)^3}$ //

(b) $y'' + y' - 2y = 18xe^x$

Put $y = e^{\lambda x}$, LHS $= \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) \Rightarrow \lambda = 1$ or -2

[5]

$\Rightarrow$ C.F. $= c_1 e^x + c_2 e^{-2x}$ //

For PI, since e^x is part of PI

$\Rightarrow$ Try $y = x(Ax+B)e^x$

$y'' + y' - 2y = e^x(6Ax + 3B + 2A) \overset{!}{=} 18xe^x \Rightarrow A = 3$, $B = -2$

$\therefore y_{PI} = x(3x-2)e^x$

[10]

$\therefore$ general soln $y = c_1 e^x + c_2 e^{-2x} + (3x^2 - 2x)e^x$ //

2. $y'' - y = \operatorname{sech} x$

Homogeneous eqn $y'' - y = 0 \Rightarrow y = \sinh x$ or $\cosh x$ [5]

variation of parameters: $y = A(x) \sinh x + B(x) \cosh x$

$$y' = \underbrace{A' \sinh x + B' \cosh x}_{\overset{!}{=} 0 \text{ required}} + A \cosh x + B \sinh x$$

$$y'' = A' \cosh x + B' \sinh x + \underbrace{A \sinh x + B \cosh x}_{y}$$

$$\therefore y'' - y = A' \cosh x + B' \sinh x \overset{!}{=} \operatorname{sech} x$$

$\Rightarrow$ solve $\begin{cases} A' \sinh x + B' \cosh x = 0 \\ A' \cosh x + B' \sinh x = \operatorname{sech} x \end{cases}$ [10]

$$\Rightarrow \begin{aligned} A' &= \cosh x \operatorname{sech} x = 1 \\ B' &= -\sinh x \operatorname{sech} x = -\tanh x \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= x \\ B &= -\ln \cosh x \end{aligned}$$

$$\therefore y = x \sinh x - (\ln \cosh x) \cosh x + c_1 \sinh x + c_2 \cosh x.$$ [10]

3.
(a) $\cancel{\phi}\, y'' = f(x)$

Write $y = \int_0^1 G(x,t)\, f(t)\, dt$

G has to satisfy : $\begin{cases} G''(x,t) = \delta(x-t) & \text{— ①} \\ G(0,t) = 0 & \text{— ②} \\ G'(1,t) = 0 & \text{— ③} \end{cases}$ [3]

For $t \in (0,1)$, $\frac{d^2 G}{dx^2} = 0 \Rightarrow G(x,t) = \begin{cases} A_1 x + A_2 & x < t \\ B_1 x + B_2 & x > t \end{cases}$ [2]
$\quad \ \ \neq$
$\quad \ \ x$

② $\Rightarrow$ $G(0,t) = 0 \Rightarrow A_2 = 0 \Rightarrow G = A_1 x$, $x < t$

③ $\Rightarrow$ $G'(1,t) = 0 \Rightarrow B_1 = 0 \Rightarrow G = B_2$, $x > t$ [2]

Finally ① $\Rightarrow$ $\int_0^1 G''(x,t)\, dx = \int \delta(x-t)\, d\,\text{--}\ 1$

$$\begin{aligned} &\ \ \parallel \\ &G'(t+\varepsilon, t) - G'(t-\varepsilon, t) \\ &\ \ \parallel \\ &0 - A_1 \end{aligned} \qquad \Rightarrow A_1 = -1$$

Continuity of G at $x = t \Rightarrow A_1 t = B_2 \Rightarrow B_2 = -t$ [4]

$\therefore\ G(x,t) = \begin{cases} -x & , \ x < t \\ -t & , \ x > t \end{cases}$ //

Finally, $y(x) = \int_0^1 G(x,t)\, f(t)\, dt = \int_0^x (-t)\, f(t)\, dt + \int_x^1 -x\, f(t)\, dt$

$= -\int_0^x t\, f(t)\, dt \quad - x \int_x^1 f(t)\, dt$ // [2]

(b) $y'' - y = f(x)$

$$y = \int_{-\infty}^{\infty} G(x,t)\, f(t)\, dt$$

G has to satisfy : $\begin{cases} G'' - G = \delta(x-t) \\ G(-\infty, t) = 0 \\ G(\infty, t) = 0 \end{cases}$ [3]

For $t \neq x$, $G'' - G = 0 \Rightarrow G(x,t) = \begin{cases} A_1 e^{-x} + A_2 e^{x} & x < t \\ B_1 e^{-x} + B_2 e^{x} & x > t \end{cases}$ [2]

B.C. $G(-\infty, t) = 0 \Rightarrow A_1 = 0$ [2]

$G(\infty, t) = 0 \Rightarrow B_2 = 0$

Continuity : $\begin{cases} -B_1 e^{-t} - A_2 e^{t} = 1 \\ B_1 e^{-t} = A_2 e^{t} \end{cases}$

$\Rightarrow B_1 = -\frac{1}{2} e^{t}, \quad A_2 = -\frac{1}{2} e^{-t}$

$\therefore G(x,t) = \begin{cases} -\frac{1}{2} e^{x-t} & x < t \\ -\frac{1}{2} e^{t-x} & x > t \end{cases} = -\frac{1}{2} e^{-|x-t|}$

[4]

$$y = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-t|} f(t)\, dt$$ [1]

//

4. (a) $z=0$ is a regular singular point since

$$y'' + \frac{1}{z}y' + \frac{z-1}{z^2}y = 0$$

$$\lim_{z=0} z\,p(z) = 1$$

$$\lim_{z=0} z^2 q(z) = -1$$

[4]

(b) Consider $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$

then $y' = \sum (n+\sigma) a_n z^{n+\sigma-1}$

$y'' = \sum (n+\sigma)(n+\sigma-1) a_n z^{n+\sigma-2}$

sub. into DE, $\sum_{n=0} (n+\sigma)(n+\sigma-1) a_n z^{n+\sigma} + \sum_{n=0} (n+\sigma) a_n z^{n+\sigma}$

$$+ \underbrace{\sum_{n=0} a_n z^{n+\sigma+1}}_{\sum_{n=1} a_{n-1} z^{n+\sigma}} - \sum_{n=0} a_n z^{n+\sigma} = 0$$

$\therefore$ $n=0$ term: $(\underbrace{\sigma(\sigma-1) + \sigma - 1}_{(\sigma+1)(\sigma-1)}) a_0 z^\sigma \stackrel{!}{=} 0 \Rightarrow \sigma = \pm 1$ [4]

$n \geq 1$ term: $[\underbrace{(n+\sigma)(n+\sigma-1) + (n+\sigma) - 1}_{(n+\sigma+1)(n+\sigma-1)}] a_n + a_{n-1} = 0$ [4]

For $\sigma = 1$, $a_n = -\dfrac{a_{n-1}}{(n+2)n}$

Solving, $a_n = \dfrac{(-1)^n 2 a_0}{n!\,(n+2)!}$

$\therefore y_1 = a_0 z\left[1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n+2)!} z^n\right]$ [3]

(c) Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W' = y_1 y_2'' - y_2 y_1'' = -pW \qquad\qquad y'' + py' + qy = 0$$

$$\Rightarrow \quad W = c \cdot \exp\left(-\int^{z} p(u)\, du\right)$$

For our case, $p = 1/z$, $W = c/z$ [3]

$$\therefore \quad y_1 y_2' - y_2 y_1' = c/z$$

$$\| $$

$$y_1^2 \left(\frac{y_2}{y_1}\right)'$$

$$\Rightarrow \left(\frac{y_2}{y_1}\right)' = c\,\frac{1}{z\, y_1^2(z)} \quad \Rightarrow \quad y_2 = y(z)\int^{z} du\, \frac{c}{u\, y_1^2(u)}$$ [3]

Now $y_1(z) = z - \frac{z^2}{3} + \frac{z^3}{24} - \frac{z^4}{360} + \cdots$

$$y_1^2 = z^2 - \frac{1}{3}\times 2\, z^3 + \left(\frac{1}{24}\times 2 + \frac{1}{9}\right) z^4 + \cdots$$

$$= z^2 - \frac{2}{3} z^3 + \frac{7}{36} z^4 + \cdots$$

$$z y_1^2 = z^3 \cdot \left(1 - \frac{2}{3} z + \frac{7}{36} z^2 + \cdots\right)$$

$$\frac{1}{z y_1^2} = \frac{1}{z^3}\left(\frac{1}{1 - \frac{2}{3} z + \frac{7}{36} z^2 + \cdots}\right)$$

$$= \frac{1}{z^3}\left(1 + \frac{2}{3} z + \left(\frac{4}{9} - \frac{7}{36}\right) z^2 + \cdots\right)$$

$$= \frac{1}{z^3}\left(1 + \frac{2}{3} z + \frac{1}{4} z^2 + \cdots\right)$$

$$\int^{z} du\, \frac{1}{u\, y_1^2(u)} = -\frac{1}{2z^2} - \frac{2}{3z} + \frac{1}{4}\ln z + O(z)$$

$$\therefore\ y_2 = y_1 \cdot \left[\frac{1}{4}\ln z - \frac{1}{2z^2} - \frac{2}{3z} + O(z)\right] c.$$

$$= \frac{c}{4}\ln z \cdot y_1(z) + c\underbrace{\left(z - \frac{z^2}{3} + \frac{z^3}{24} + \cdots\right)\left(-\frac{1}{2z^2} - \frac{2}{3z} + O(z)\right)}_{-\frac{1}{2z} - \frac{2}{3} + \frac{1}{6} + \left[\left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right) + \frac{1}{24}\left(-\frac{1}{2}\right)\right]z + O(z^2)}$$

$$= -\frac{1}{2z} - \frac{1}{2} + \frac{29}{144}z + O(z^2)$$

[4]