1 (a) operator $\mathcal{L}$ is self adjoint if $\mathcal{L} = \mathcal{L}^+$ where $\mathcal{L}^+$ is defined by

$$\int_a^b f^* \mathcal{L} g \, dx = \int_a^b (\mathcal{L}^+ f)^* g \, dx + \text{boundary terms}$$

Hermitian if self adjoint + boundary terms vanish. [5]

Now $\int_a^b f^* \left(\frac{d}{dx} p \frac{d}{dx} g + q g\right) = \int_a^b \frac{d}{dx}\left(f^* p \frac{d}{dx} g\right) - \int_a^b \frac{df^*}{dx} \cdot p \frac{dg}{dx} + \int_a^b f^* q g$

$$= f^* p g' \Big|_a^b - g p f^{*\prime} \Big|_a^b + \int_a^b \frac{d}{dx}\left(p \frac{df^*}{dx}\right) g + \int_a^b \underbrace{q f^*}_{= (q^* f)^*} g$$

$$\therefore \mathcal{L}^+ = \frac{d}{dx}\left(p^* \frac{d}{dx}\right) + q^*$$ [5]

$\mathcal{L}$ is self adjoint if $p = p^*$, $q = q^*$ real.

Hermitian if p, q real and $f^* p g' - (f^*)' p g \Big|_a^b = 0$. [5]

In particular Boundary terms $= 0$ for $y(a) = y(b) = 0$. [2]

(b)

$\mathcal{L} y_i = \lambda_i \rho y_i$ $\quad\quad \rho = \text{real}$

$\mathcal{L} y_j = \lambda_j \rho y_j$

$\Rightarrow \quad y_j^* \mathcal{L} y_i - y_i (\mathcal{L} y_j)^* = (\lambda_i - \lambda_j^*) y_j^* \rho y_i$

$$\underbrace{\int_a^b y_j^* \mathcal{L} y_i - \int y_i \underbrace{(\mathcal{L} y_j)^*}_{\text{ii. Hermitian } (\mathcal{L}^+ y_j)^*}}_{0} = (\lambda_i - \lambda_j^*) \int y_j^* \rho y_i \quad \Rightarrow \text{result}$$ [8]

2.(a) $N_0 = 0$, $N_1 = \infty$ [2]

$$\frac{d}{dt} G(x,t) = (-2t + 2x)\, G = \sum_{n=0} H_n \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} H_{n+1} \frac{t^n}{n!}$$

$$= \underbrace{-2\sum_{n=0}^{\infty} H_n \frac{t^{n+1}}{n!}}_{= -2\sum_{n=1}^{\infty} n H_{n-1} \frac{t^n}{n!}} + 2x \sum_{n=0}^{\infty} H_n \frac{t^n}{n!}$$

$$\Rightarrow \quad H_{n+1} = 2x H_n - 2n H_{n-1} \qquad — (1)$$ [5]

Now if H_n is a polynomial of x of order n for all $n \leq N$

then H_{N+1} is also so due to (1).

Also, $H_0 = 1$ is a poly. of x of order 0.

So by mathematical induction, the statement is true [5]

(b) $y'' - 2xy' + 2ny = 0$

$$(y e^{-x^2})' + 2n e^{-x^2} y = 0$$

$$p = e^{-x^2}, \quad \rho = e^{-x^2}$$ [5]

$$\therefore \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 0 \quad \text{for } n \neq m$$ [3]

Now, $$\int_{-\infty}^{\infty} e^{-t^2+2tx} e^{-t^2+2tx} e^{-x^2} dx = \sum_{n,m} \frac{1}{n!}\frac{1}{m!} t^{n+m} \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx$$

$$= e^{-2t^2} \int_{-\infty}^{\infty} \underbrace{e^{-x^2+4tx}}_{= e^{-(x-2t)^2+4t^2}} dx = e^{2t^2}\sqrt{\pi} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{2^k t^{2k}}{k!}$$

$\Rightarrow$ required result [5]

3 (a) $t = ie^{i\theta}$

$$\frac{1}{2}(t - t^{-1}) = \frac{i}{2}(e^{i\theta} + e^{-i\theta}) = i\cos\theta$$

$$\therefore e^{iz\cos\theta} = \sum_n J_n(z)\, i^n e^{in\theta}$$

$J_{-n}(z) = (-1)^n J_n(z)$ [5]

$$= J_0(z) + \sum_{n=1}^{\infty} J_n(z)\, i^n e^{in\theta} + \sum_{n=1}^{\infty} J_n(z) \underbrace{(-1)^n\, i^{-n}}_{i^n} e^{-in\theta}$$

$$= J_0(z) + \sum_{n=1}^{\infty} J_n(z)\, i^n\, 2\cos n\theta$$

real part: $\cos(z\cos\theta) = J_0(z) + \sum_{n=1}^{\infty} J_{2n}(z)(-1)^n \cos 2n\theta$ [5]

(b) ~~Let $\theta = \pi/2$, $1 = J_0 + 2\sum_{n=1}^{\infty} J_{2n}$~~

change $t \to -t$, $e^{-\frac{z}{2}(t - t^{-1})} = \sum J_n (-1)^n t^n$

$$e^{\frac{z}{2}(t - t^{-1})} = \sum J_n\, t^n$$

multiply $1 = \sum J_n(z) J_m(z) (-1)^n t^{n+m}$

$$= \sum_{n=-\infty}^{\infty} t^n \sum_{m=-\infty}^{\infty} (-1)^m J_m(z) J_{n-m}(z)$$ [5]

$n=0$: $\sum_{m=-\infty}^{\infty} J_m(z) \underbrace{(-1)^m J_{-m}(z)}_{J_m(z)} = 1$ [2]

$2n$: $\sum_{m=-\infty}^{\infty} (-1)^m J_m(z) J_{2n-m}(z) = 0$ [3]

(c) Put $z = x+y$

$$e^{+\frac{z}{2}(t-t^{-1})} = e^{\frac{x}{2}(t-t^{-1})} \, e^{\frac{y}{2}(t-t^{-1})}$$

$$\| \qquad = \sum_{n=-\infty}^{\infty} t^n \sum_{k=-\infty}^{\infty} J_k J_{n-k}$$

$$\sum_{n=-\infty}^{\infty} J_n(x+y)\, t^n$$

$$\Rightarrow J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x)\, J_{n-k}(y)$$ //

[5]

4. (a) $\int \phi\, \phi'' + \int (\lambda - x^2)\, \phi^2 = 0$

$$\Rightarrow \underbrace{\lambda \int_0^1 \phi^2\, dx}_{\text{positive}} = \int_0^1 x^2 \phi^2 \; \cancel{+} - \underbrace{\int_0^1 \phi\, \phi''}_{-\int_0^1 (\phi\phi')' + \int_0^1 (\phi')^2 \; = \; \int_0^1 (\phi')^2} = \int_0^1 x^2 \phi^2 + \int_0^1 (\phi')^2$$

positive

$\Rightarrow \lambda \geq 0$ //

[15]

(b) If $\lambda = 0$, then $\int_0^1 x^2 \phi^2 = 0 = \int_0^1 (\phi')^2$

$$\Rightarrow \begin{cases} x\phi = 0 \\ \phi' = 0 \end{cases}$$ almost everywhere $\Rightarrow$ $\phi = 0$ almost everywhere

$\Downarrow$

ϕ = constant

$\therefore \lambda \neq 0$ //

[10]