

The abelian and non-abelian Josephson  
effect and pseudo-Goldstone bosons

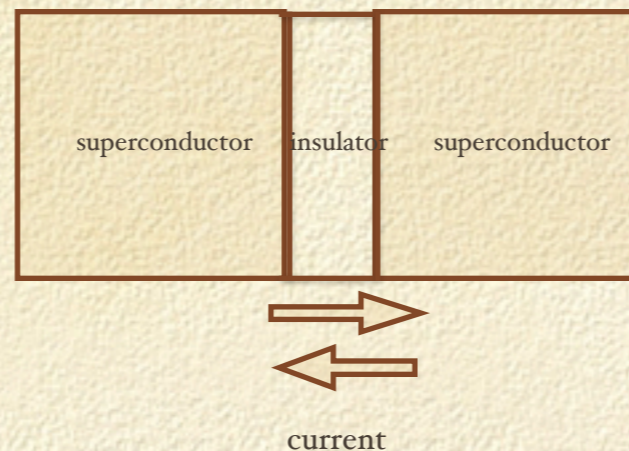
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# The Josephson Effect

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- Two samples of a superconductor are brought into close together with a thin layer of insulator between them.
- This allows for an overlap of the wave functions of the Cooper pairs in each sample, and they are able to interact with one another.
- This interaction allows for tunneling between the the superconductors, across the junction.



# Feynman description of the Josephson effect

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- The conducting electron fluid is in one macroscopic quantum state for each superconductor.
- Only one complex number, its phase and amplitude effectively and adequately describes the behavior of each superconductor
- The problem reduces to the interaction of these effective degrees of freedom.

□ The Hamiltonian is given by: 
$$\mathcal{H} = \begin{pmatrix} E_L & K \\ K & E_R \end{pmatrix}$$

- With the corresponding Schrödinger equation:

$$i\hbar\partial_t\psi(t) = E_L\psi(t) + K\chi(t)$$

$$i\hbar\partial_t\chi(t) = E_R\chi(t) + K\psi(t)$$



- For the DC Josephson effect,  $E_L = E_R \equiv E$  and the system has the solution:

$$\begin{pmatrix} \psi(t) \\ \chi(t) \end{pmatrix} = e^{-iEt/\hbar} \left( \cos(-Kt/\hbar) + i \sin(-Kt/\hbar) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} \psi_0 \\ \chi_0 \end{pmatrix}$$

- The conserved charge is given by:

$$Q = \psi^*(t)\psi(t) + \chi^*(t)\chi(t) \quad \dot{Q} = 0$$

- However the charge on the left and right are given by:

$$Q_\psi = \psi^*(t)\psi(t) \quad Q_\chi = \chi^*(t)\chi(t)$$

$$\dot{Q}_\psi = -\dot{Q}_\chi$$



□ Thus the charge would slosh back and forth between the two sides.

Replacing

$$\psi_0 = \sqrt{\rho} e^{i\theta_\psi} \quad \chi_0 = \sqrt{\rho} e^{i\theta_\chi}$$

gives  $Q_\psi = \rho (1 - \sin((2K/\hbar)t) \sin(\theta_\psi - \theta_\chi))$

The result is usually quoted only for the initial current

$$\dot{Q}_\psi = \rho (2K/\hbar \sin(\theta_\chi - \theta_\psi))$$

which is the familiar result for the Josephson current.

For the AC Josephson effect, we replace

$$E_L \rightarrow E + V \quad E_R \rightarrow E - V$$

and for  $K \ll V$  one easily finds

$$Q_\psi = \rho \left( 1 - \frac{K}{V} \cos(2Vt/\hbar + (\theta_\chi - \theta_\psi)) \right)$$

$$\dot{Q}_\psi = \rho \frac{2K}{\hbar} \sin(2Vt/\hbar + (\theta_\chi - \theta_\psi))$$

The Josephson acceleration equation follows straightforwardly from the equations of motion for the time dependent phases:

$$\psi(t) = \sqrt{\rho_\psi(t)} e^{i\theta_\psi(t)} \quad \chi = \sqrt{\rho_\chi(t)} e^{i\theta_\chi(t)}$$
$$\dot{\theta}_\chi(t) - \dot{\theta}_\psi(t) = 2V/\hbar$$



How can we obtain this from an effective Lagrangian point of view?

$$\mathcal{L} = \psi^\dagger i\hbar\dot{\psi} + \chi^\dagger i\hbar\dot{\chi} - (\psi^\dagger \chi^\dagger) \begin{pmatrix} E + V & 0 \\ 0 & E - V \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} - (\psi^\dagger \chi^\dagger) \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

Each field is not a wave function, but a quantum field. In the absence of the coupling term,  $K = 0$ , the symmetry of this model corresponds to independent phase transformations of the two fields  $\psi \rightarrow e^{i\zeta}\psi$  and  $\chi \rightarrow e^{i\eta}\chi$ .

The fact that physically the amplitudes, on either side, of the effective fields are non-zero and vary very little, means that the  $U(1)$  symmetry is spontaneously broken. The photon, which we have not included in our analysis, will then absorb the attendant Goldstone boson and become massive giving the Meissner effect.

This does not happen independently on either side of the junction. There really are not two independent photons. The true symmetry of the theory is not  $U(1) \times U(1)$  but the diagonal subgroup  $U_D(1)$ , thus only one photon arises and one Goldstone boson.

However the coupling explicitly breaks the symmetry from  $U(1) \times U(1)$  to  $U_D(1)$ .

Explicit but soft breaking of otherwise spontaneously broken symmetries give rise to light almost massless particles, called pseudo-Goldstone bosons. The coupling constant  $K$  is the soft breaking parameter. A phase transformation that is equal but opposite on either side of the junction corresponds to excitations of the pseudo-Goldstone bosons. The frequency associated with these oscillations is correspondingly small  $\omega = \frac{K}{\hbar}$ .



# The non-abelian Josephson effect

- Can be formulated as a junction of two effective systems with a weak interaction between them.
- Each system should have the same symmetry group.
- The symmetry should be spontaneously broken on each side. This would give a doubling of Goldstone modes, which is an artifact of the description.
- The coupling of the systems together should preserve only the diagonal symmetry, hence giving rise to pseudo-Goldstone modes for the explicitly broken symmetries.
- Excitations of these modes will correspond to the (non-abelian) generalization of the Josephson effect.



# $SO(3)$ model

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- We consider a model with a doublet on one side and a triplet on the other.

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_{\vec{\phi}} + \mathcal{L}_I$$

$$\mathcal{L}_\psi = \dot{\psi}^\dagger \dot{\psi} - \lambda(\psi^\dagger \psi - a^2)^2$$

$$\mathcal{L}_{\vec{\phi}} = \frac{1}{2} \dot{\vec{\phi}} \cdot \dot{\vec{\phi}} - \gamma(\vec{\phi} \cdot \vec{\phi} - v^2)^2$$

$$\mathcal{L}_I = -K \psi^\dagger \vec{\sigma} \psi \cdot \vec{\phi}$$



- The symmetry for the uncoupled systems is:

$$U(2) \times SO(3)$$

- This spontaneously breaks to:

$$U(1) \times SO(2)$$

- The coupling explicitly breaks the symmetry to:

$$U(1) \times SO_D(3)$$

- Thus one expects only three Goldstone boson which are eaten and two pseudo-Goldstone mode giving a Josephson effect



- We must find the spectrum of small oscillations around the minimum of the potential.

$$2\lambda(\psi^\dagger\psi - a^2)\psi + K\vec{\sigma}\psi \cdot \vec{\phi} = 0$$

$$4\gamma(\vec{\phi} \cdot \vec{\phi} - v^2)\vec{\phi} + K\psi^\dagger\vec{\sigma}\psi = 0$$

- We take  $\psi = \psi_R \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  which implies  $\phi = \phi_3(0, 0, 1)$

- This yields the equations:

$$2\lambda(\psi_R^2 - a^2)\psi_R - K\psi_R\phi_3 = 0$$

$$4\gamma(\phi_3^2 - v^2)\phi_3 - K\psi_R^2 = 0.$$



- The solution is possible in terms of a cubic equation, however a perturbative solution suffices, to first order in the coupling:

$$\psi_R = a + \frac{a^2}{8\gamma v^2} K$$

$$\phi_3 = v + \frac{v}{4\lambda a} K.$$

- This minimum is inserted into the matrix of second derivatives giving a  $7 \times 7$  :

$$\frac{\partial^2 V}{\partial f_i \partial f_j} = \begin{pmatrix} 4Kv & 0 & 0 & 0 & 2Ka & 0 & 0 \\ 0 & 4Kv & 0 & 0 & 0 & -2Ka & 0 \\ 0 & 0 & 8\lambda a^2 + 4vK & 0 & 0 & 0 & -2Ka \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2Ka & 0 & 0 & 0 & Ka^2/v & 0 & 0 \\ 0 & -2Ka & 0 & 0 & 0 & Ka^2/v & 0 \\ 0 & 0 & -2Ka & 0 & 0 & 0 & 8\gamma v^2 + 3Ka^2/v \end{pmatrix}$$



- Diagonalizing gives the spectrum of frequencies and the corresponding eigenvectors:

$$\omega = 0; \psi_{2I} \sim v_1$$

$$\omega = 0; (\psi_{1R}, \phi_1) = (a, -2v) \sim v_2$$

$$\omega = (4v + a^2/v)K; (\psi_{1R}, \phi_1) = (2v, a) \sim v_3$$

$$\omega = 0; (\psi_{1I}, \phi_2) = (a, 2v) \sim v_4$$

$$\omega = (4v + a^2/v)K; (\psi_{1I}, \phi_2) = (-2v, a) \sim v_5$$

$$\omega = 8\lambda a^2; (\psi_{2R}, \phi_3) = (1, 0) \sim v_6$$

$$\omega = 8\gamma v^2; (\psi_{2R}, \phi_3) = (0, 1) \sim v_7$$

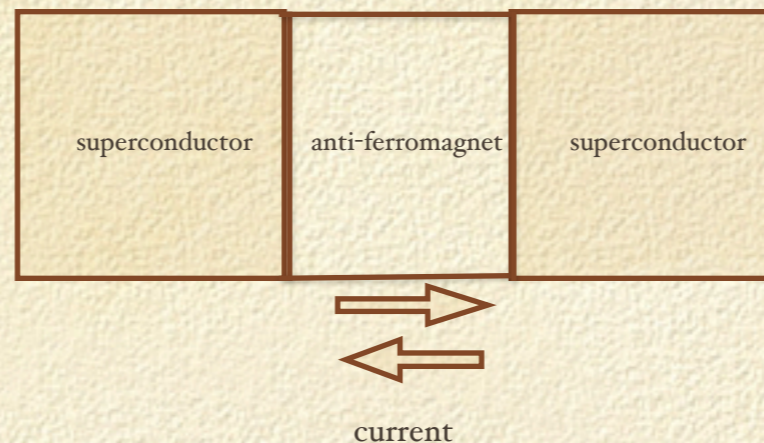


- The last two modes decouple in the limit that
$$\lambda, \gamma \rightarrow \infty$$
- The uncoupled system is expected to have 5 Goldstone bosons.
- The coupling explicitly breaks two of the symmetries, hence we expect three Goldstone bosons and two pseudo-Goldstone bosons, exactly as we have found.
- The two symmetries that leave the minimum invariant are not spontaneously broken and do not give rise to light or massless excitations.



# $SO(5)$ model

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- The complex of high  $T_c$  superconductivity and anti-ferromagnetism in doped CuO systems has been conjectured to be described by an  $SO(5)$  invariant effective theory.
- We study the question of non-abelian Josephson junctions in these materials.



The effective Lagrangian is given by in terms of a five component scalar field  $\vec{\varphi}$  where the first two components correspond to the superconducting order parameter  $\vec{\phi} \equiv (\varphi_1, \varphi_2)$  while the latter three correspond to the anti-ferromagnetic order parameter  $\vec{n} \equiv (\varphi_1, \varphi_2, \varphi_3)$ . The effective Lagrangian for the  $SO(5)$  invariant critical theory is given by:

$$\mathcal{L}_{SO(5)}(\vec{\varphi}) = \frac{1}{2} \dot{\vec{\varphi}} \cdot \dot{\vec{\varphi}} - \lambda (\vec{\varphi} \cdot \vec{\varphi} - a^2)^2$$

The symmetry is explicitly broken by the doping which pushes the minimum to be in the superconducting phase or the anti-ferromagnetic phase. The residual symmetry is  $SO(3) \times SO(2)$

$$\mathcal{L}_{\text{doping}}(g, \vec{\varphi}) = -g(\vec{\phi} \cdot \vec{\phi} - \vec{n} \cdot \vec{n})$$



For positive coupling the system is anti-ferromagnetic, for negative it is superconducting.

For the critical theory  $SO(5) \rightarrow SO(4)$  which has 4 spontaneously broken directions and hence 4 Goldstone bosons.

With the doping, for the superconductivity, 3 of these generators are explicitly broken, leaving one Goldstone boson and three pseudo-Goldstone bosons.

For the anti-ferromagnet, two generators are spontaneously broken and give rise to Goldstone bosons while two are explicitly broken giving two pseudo-Goldstone modes.

These modes have nothing to do with the Josephson effect.



For a junction we couple two such systems together, with the simplest interaction that preserves only the diagonal  $SO(3) \times SO(2)$  symmetry. We take

$$\mathcal{L}_{\text{Josephson}} = \mathcal{L}_{SO(5)}(\vec{\varphi}_1) + \mathcal{L}_{\text{doping}}(g_1, \vec{\varphi}_1) + \mathcal{L}_{SO(5)}(\vec{\varphi}_2) + \mathcal{L}_{\text{doping}}(g_2, \vec{\varphi}_2) + K \vec{\varphi}_1 \cdot \vec{\varphi}_2$$

We are interested in finding new forms of Josephson effects, hence we are not interested to describe tunneling within the superconducting subsystem or even the anti-ferromagnetic subsystem but between the two systems. Hence we replace each order parameter with its amplitude:  $|\vec{\phi}_1| = \phi_1$   $|\vec{\phi}_2| = \phi_2$   $|\vec{n}_1| = n_1$   $|\vec{n}_2| = n_2$

The reduced effective Lagrangian is then given by:

$$\begin{aligned} \mathcal{L}_{\text{Josephson, reduced}} &= \frac{1}{2}(\dot{\phi}_1^2 + \dot{n}_1^2) - \lambda(\phi_1^2 + n_1^2 - a^2)^2 + \frac{1}{2}(\dot{\phi}_2^2 + \dot{n}_2^2) - \lambda(\phi_2^2 + n_2^2 - a^2)^2 \\ &- g_1(\phi_1^2 - n_1^2) + g_2(\phi_2^2 - n_2^2) \\ &+ K(\phi_1\phi_2 + n_1n_2) \end{aligned}$$



The equations for the minimum are:

$$(4\lambda(\phi_1^2 + n_1^2 - a^2) + 2g_1)\phi_1 - K\phi_2 = 0$$

$$(4\lambda(\phi_2^2 + n_2^2 - a^2) - 2g_2)\phi_2 - K\phi_1 = 0$$

$$(4\lambda(\phi_1^2 + n_1^2 - a^2) - 2g_1)n_1 - Kn_2 = 0$$

$$(4\lambda(\phi_2^2 + n_2^2 - a^2) + 2g_2)n_2 - Kn_1 = 0$$

This system of equations, although quite non-linear can be exactly and analytically solved.

However we are always interested in small coupling we will freely use perturbation theory.

These equations are equivalently written as:

$$\begin{pmatrix} A + 2g_1 & -K \\ -K & B - 2g_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} A - 2g_1 & -K \\ -K & B + 2g_2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = 0$$

with  $A = 4\lambda(\phi_1^2 + n_1^2 - a^2)$        $B = 4\lambda(\phi_2^2 + n_2^2 - a^2)$



For there to be a non-trivial solution, the determinant must vanish for each pair

$$(A + 2g_1)(B - 2g_2) - K^2 = 0$$

$$(A - 2g_1)(B + 2g_2) - K^2 = 0$$

With solution:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \pm \sqrt{4 + (K^2/g_1g_2)} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

This solution is independent of  $\lambda$  especially in the limit that it is infinite, which decouples the radial modes. Then the solution is:

$$\phi_1 = \frac{K}{A + 2g_1} \phi_2$$

$$n_1 = \frac{K}{A - 2g_1} n_2$$



With  $\phi_2$  and  $n_2$  determined by self consistency:

$$A = 4\lambda \left( \left( \frac{K}{A+2g_1} \phi_2 \right)^2 + \left( \frac{K}{A-2g_1} n_2 \right)^2 - a^2 \right) = \sqrt{4 + (K^2/g_1g_2)}g_1$$

$$B = 4\lambda(\phi_2^2 + n_2^2 - a^2) = \sqrt{4 + (K^2/g_1g_2)}g_2$$

This system is linear in  $\phi_2^2$  and  $n_2^2$  which is easily solved and gives (perturbative):

$$\phi_1 = \frac{K}{A+2g_1} \sqrt{\frac{a^2 K^2 g_1}{8Ag_2^2} \left( \left( \frac{K}{A-2g_1} \right)^2 - 1 \right)}$$

$$n_1 = \frac{K}{A-2g_1} \sqrt{\frac{a^2 K^2 g_1}{8Ag_2^2} \left( 1 - \left( \frac{K}{A+2g_1} \right)^2 \right)}$$

$$\phi_2 = \sqrt{\frac{a^2 K^2 g_1}{8Ag_2^2} \left( \left( \frac{K}{A-2g_1} \right)^2 - 1 \right)}$$

$$n_2 = \sqrt{\frac{a^2 K^2 g_1}{8Ag_2^2} \left( 1 - \left( \frac{K}{A+2g_1} \right)^2 \right)}.$$



It is easy to see that the minimum is largely in the anti-ferromagnetic direction for one side and largely superconducting for the other side. However the coupling does add a little mixture to the minima.

This will allow for a weak regular Josephson effect between the two sides if we allow for a direction in the superconducting order parameter.

We are interested in tunneling between the superconductor and the anti-ferromagnet. This requires calculating the frequency spectrum for small oscillations.



The matrix of second derivatives evaluated at the minimum is:

$$V'' = \begin{pmatrix} A + 2g_1 + 8\lambda\phi_1^2 & 8\lambda\phi_1 n_1 & -K & 0 \\ 8\lambda\phi_1 n_1 & A - 2g_1 + 8\lambda n_1^2 & 0 & -K \\ -K & 0 & B - 2g_2 + 8\lambda\phi_2^2 & 8\lambda\phi_2 n_2 \\ 0 & -K & 8\lambda\phi_2 n_2 & B + 2g_2 + 8\lambda n_2^2 \end{pmatrix}$$

This can be exactly diagonalized, but that is not very illuminating. A better approach is perturbation. There are two very heavy modes and two light modes. If the coupling and symmetry breaking is zero, the light modes are massless and degenerate. We can find the light frequencies in degenerate perturbation theory.

$$V'' = V_0'' + V_1''$$



with:

$$V_0'' = \begin{pmatrix} 8\lambda\phi_1^2 & 8\lambda\phi_1n_1 & 0 & 0 \\ 8\lambda\phi_1n_1 & 8\lambda n_1^2 & 0 & 0 \\ 0 & 0 & 8\lambda\phi_2^2 & 8\lambda\phi_2n_2 \\ 0 & 0 & 8\lambda\phi_2n_2 & 8\lambda n_2^2 \end{pmatrix}$$

$$V_1'' = \begin{pmatrix} A + 2g_1 & 0 & -K & 0 \\ 0 & A - 2g_1 & 0 & -K \\ -K & 0 & B - 2g_2 & 0 \\ 0 & -K & 0 & B + 2g_2 \end{pmatrix}$$

Degenerate perturbation theory requires diagonalizing the perturbing matrix in the subspace of the degenerate modes.

$$v_1 = (1/a) \begin{pmatrix} n_1 \\ -\phi_1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = (1/a) \begin{pmatrix} 0 \\ 0 \\ n_2 \\ -\phi_2 \end{pmatrix}$$



This gives the 2 by 2 matrix:

$$\begin{aligned} V''_{\Delta} &= \begin{pmatrix} v_1^T V_1'' v_1 & v_1^T V_1'' v_2 \\ v_2^T V_1'' v_1 & v_2^T V_1'' v_2 \end{pmatrix} \\ &= (1/a^2) \begin{pmatrix} n_1^2(A + 2g_1) + \phi_1^2(A - 2g_1) & -K(n_1n_2 + \phi_1\phi_2) \\ -K(n_1n_2 + \phi_1\phi_2) & n_2^2(B - 2g_2) + \phi_2^2(B + 2g_2) \end{pmatrix} \end{aligned}$$

The eigenfrequencies are:  $\omega^2 = \alpha \pm \sqrt{\beta^2 + \gamma^2}$

$$\alpha = (1/2a^2) (n_1^2(A + 2g_1) + \phi_1^2(A - 2g_1) + n_2^2(B - 2g_2) + \phi_2^2(B + 2g_2))$$

$$\beta = (1/2a^2) (n_1^2(A + 2g_1) + \phi_1^2(A - 2g_1) - n_2^2(B - 2g_2) - \phi_2^2(B + 2g_2))$$

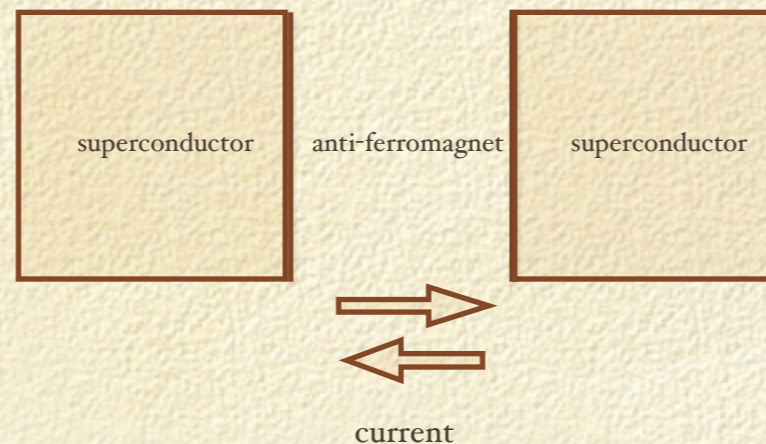
$$\gamma = -(1/2a^2)K(n_1n_2 + \phi_1\phi_2)$$

It would be nice if this current were be observed.



# The Josephson Sandwich

Consider the situation in the picture:



and the potential

$$V = \lambda(|\vec{\phi}_1|^2 - a^2)^2 + \lambda(|\vec{\phi}_2|^2 - a^2)^2 + \lambda(|\vec{\phi}_3|^2 - a^2)^2$$
$$- g_1(\varphi_1^2 + \psi_1^2 - \eta_1^2) + g_2(\varphi_2^2 + \psi_2^2 - \eta_2^2)$$
$$- g_1(\varphi_3^2 + \psi_3^2 - \eta_3^2) - K \vec{\phi}_2 \cdot (\vec{\phi}_1 + \vec{\phi}_3)$$

with:  $\vec{\phi}_i = (\varphi_i, \psi_i, \eta_i)$



We get the system:

$$\begin{pmatrix} A_\xi & -K & 0 \\ -K & B_\xi & -K \\ 0 & -K & A_\xi \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0,$$

Where:

$$\xi = \varphi$$

$$\xi = \eta$$

$$A_\varphi = \alpha - 2g_1$$

$$A_\eta = \alpha + 2g_1$$

$$B_\varphi = \beta + 2g_2$$

$$B_\eta = \beta - 2g_2$$

and

$$\alpha = 4\lambda(|\vec{\phi}_1|^2 - a^2) \quad \beta = 4\lambda(|\vec{\phi}_2|^2 - a^2)$$



The systems are exactly solvable, however a perturbative approach is sufficient.

The determinant of each system must vanish yielding:

$$A_{\varphi}(A_{\varphi}B_{\varphi} - 2K^2) = 0$$

$$A_{\eta}(A_{\eta}B_{\eta} - 2K^2) = 0$$

With solution:

$$\alpha = \sqrt{\frac{2g_1}{g_2}(K^2 + 2g_1g_2)}, \quad \beta = (g_2/g_1)\alpha$$



This can be used, in conjunction with the definitions of  $\alpha$  and  $\beta$  to find the values of  $|\vec{\phi}_1|^2$  and  $|\vec{\phi}_3|^2$  in terms of the coupling constants. We use the diagonal  $SO(3)$  symmetry to rotate the solution into the direction with  $\psi_i = 0$  without loss of generality. Then the spectrum of oscillations are governed by the matrix of second derivatives of the potential evaluated at the position of the minimum. This is a  $9 \times 9$  matrix which we will only treat perturbatively in  $K$ ,  $g_1$  and  $g_2$ . The order zero matrix is block diagonal with three,  $3 \times 3$  blocks, of the form

$$8\lambda \begin{pmatrix} \varphi_i^2 & 0 & \varphi_i \eta_i \\ 0 & 0 & 0 \\ \varphi_i \eta_i & 0 & \eta_i^2 \end{pmatrix}$$

Which gives three heavy modes and 6 light modes. The normalized light modes are given as:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes \hat{n}_i, \quad \begin{pmatrix} \eta_i / \sqrt{\varphi_i^2 + \eta_i^2} \\ 0 \\ -\varphi_i / \sqrt{\varphi_i^2 + \eta_i^2} \end{pmatrix} \otimes \hat{n}_i$$



The 6x6 matrix of the degenerate subspace separates into two 3x3 blocks which are identical in form to the 3x3 matrix that appeared before. The eigenvectors and eigenvalues are:

$$v_{1,1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_{1,2} = \begin{pmatrix} A_\varphi \\ -2K \\ A_\varphi \end{pmatrix}, v_{1,3} = \begin{pmatrix} K \\ A_\varphi \\ K \end{pmatrix}$$

$$A_\varphi, A_\varphi + B_\varphi, 0$$

$$v_{2,1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_{2,2} = \begin{pmatrix} \alpha \\ -2K' \\ \alpha \end{pmatrix}, v_{2,3} = \begin{pmatrix} K' \\ \alpha \\ K' \end{pmatrix}$$

$$\alpha, (\alpha \pm \sqrt{(\alpha - \beta)^2 + 8(K')^2})/2$$



The “Josephson effect” corresponds to the mode  $\nu_{1,1}$ ,  $\nu_{1,2}$  corresponds to charge exchange with the intermediate anti-ferromagnet and the end superconductors, while  $\nu_{1,3}$  corresponds to the true Goldstone mode which gives rise to the Meissner effect and mass for the photon.

The other triplet of modes do not correspond to charge exchange but rather to fluctuations of the length of the superconducting and the anti-ferromagnetic order parameters.



## Conclusions

We have formulated the Josephson effect in the language of effective Lagrangians allowing for a generalization to non-abelian symmetries and the corresponding non-abelian Josephson effect.

We find that the Josephson effect corresponds to the excitations of pseudo-Goldstone bosons.

We considered a sandwich of three regions with an underlying, approximate  $SO(3)$  symmetry, that is explicitly broken to  $SO(2)$ . The unbroken  $SO(2)$  is gauged and superconducting in the end regions.

We find that the intermediate region mediates the exchange of charge between the two end regions, giving rise to a “Josephson effect”.



Our formalism could be applied to physical situations involving the spontaneous breaking of non-abelian gauged symmetries if an interface arises.

One promising area where this could occur is in the high density phases of QCD which are expected to occur in contiguous regions of neutron stars.

Other examples of situations which may provide a venue for our formalism include two-band superconductors, d-wave high  $T_c$  superconductors, p-wave heavy-fermion superconductors, the A phase of liquid Helium-3, and nonlinear optics, where complicated order parameters often occur.