Cosmological perturbation Theory

Pravabati Chingangbam

Korea Institute for Advanced Study, South Korea

- Inflation provides a mechanism by which widely separated directions in the sky could have been in causal contact in the early history of the universe.
- It predicts flat universe and perturbations that are adiabatic and gaussian with a nearly scale invariant power spectrum.

Plan of Talk

- Background dynamics of single scalar field inflation.
- Perturbations generated during inflation
    - Nature of the perturbations
    - Their time evolution
    - Statistical properties
- Anisotropies in Cosmic Microwave Background Radiation

Dynamical evolution of the universe

Dynamical description of the universe is based on :

▶ General theory of relativity: Einstein equation.

$$G_{\mu\nu} = 8\pi G \ T_{\mu\nu}$$

▶ The cosmological principle: space is homogeneous and isotropic. This reduces the number of dynamical metric variables to just one: the scale factor $a(t)$.

Assuming that the energy density that fills the universe behaves like a perfect fluid, $T_{\mu\nu}$ has two dynamical components :
$T_{00} = \rho$ and $T_{ii} = p$.
Thus we have three dynamical variables: $a(t)$, ρ and p.

Dynamical evolution of the universe

The equations that govern the evolution of a, ρ, p are:

- Acceleration equation (Raichaudhary equation):

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p)$$

- The fluid continuity equation:

$$\dot{\rho} + 3H(\rho + p) = 0$$

where $H \equiv \dot{a}/a$.

- Equation of state :

$$\omega = \frac{p}{\rho}$$

The above equations can be used to get the Friedmann Equation:

$$H^2 = \frac{\rho}{3}$$

What is the expansion history of the universe ?

$$a(t) = ??$$

Equivalent to asking, what is the kind of stress energy tensor that filled/fills the universe.

Expansion history of the universe

- Supernovae observations tell us the universe is undergoing accelerated expansion today.

$$a(t) \propto t^n, \quad n > 1.$$

- Formation of the structures we see today requires a period when energy density was dominated by matter.

$$a(t) \propto t^{2/3}.$$

- We know that

$$\rho_\gamma \propto \frac{1}{a^4}, \qquad \rho_m \propto \frac{1}{a^3}$$

This implies that if we go sufficiently back in time, ρ_γ must have been the dominant component.

$$a(t) \propto t^{1/2}.$$

▶ The CMB is very isotropic and homogeneous, with tiny fluctuations imposed on them, on scales which could not have been causally connected at any past epoch if the universe was matter or radiation dominated.

▶ If there was

$$\ddot{a} > 0$$

prior to radiation domination, then all the observable universe today could have expanded from one causally connected region at the beginning of inflation.

Horizons and length scale

Let us define the following:

- Efolding : $N \equiv \int H dt$
- Particle horizon : $d_p = a(t) \int_{t_i}^{t} \frac{dt}{a}$
- Hubble horizon : H^{-1}

During matter or radiation dominated epoch, d_p and H^{-1} grow roughly similarly.

During inflation, d_p grows exponentially, whereas, H^{-1} is almost constant. This allows causally connected patches to grow much larger than Hubble radius.

Once inflation stops, different scales reenter the Hubble radius at different times.

How much inflation?

• Roughly, the amount of inflation should be such that the size pf the observable universe today was inside the Hubble radius at the beginning of inflation.

▶ At any two times t_1 and t_2, such that $t_2 < t_1$, the corresponding length scales are related as

$$\lambda_2 = \frac{a(t_2)}{a(t_1)} \lambda_1$$

▶ Let us denote

$$
\begin{aligned}
\lambda_0 &\equiv H_0^{-1} \quad \text{Hubble radius today} \\
\lambda_i &\equiv \text{the scale corresponding to } \lambda_0 \text{ at beginning of inflation} \\
H_i^{-1} &\equiv \text{Hubble radius at beginning of inflation} \\
t_i, t_e &\equiv \text{beginning and end of inflation}
\end{aligned}
$$

▶

$$a(t_e) = e^N a(t_i).$$

How much inflation?

- Hubble radius today and the scale from which it grew are related as

$$\lambda_0 = \frac{a(t_0)}{a(t_e)} a(t_i) \; \lambda_i$$

- We demand that

$$\lambda_i < H_i^{-1}$$

- Using $T(t) \propto 1/a(t)$,

$$\Rightarrow e^N \geq \frac{T_0}{H_0} \left(\frac{H_i}{T_{t_e}} \right)$$

Put in $T_0 = 2.725K$, $H_0 = 73 \; km/s/Mpc$ and we get

$$N \geq 70 + \ln \left(\frac{H_i}{T_{t_e}} \right)$$

Realization of Inflation : concrete model

Consider single scalar field inflation coupled minimally to gravity:

$$L = \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left(\partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$$

From gravity part we get the equations:

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{1}{6} \left(\rho + 3p \right) \\
H^2 &= \frac{\rho}{3}
\end{aligned}$$

From ϕ part we get equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

Energy and momentum density ,

$$\rho = \dot{\phi}^2 + V, \quad p = \dot{\phi}^2 - V$$

Inflation $\Rightarrow$ $\ddot{a} > 0$ $\Rightarrow$ $\rho + 3p < 0$
which is satisfied if
$$V >> \dot{\phi}^2$$

▶ Slow roll parameters:
$$\epsilon_{i+1} \equiv \frac{\dot{\epsilon_i}}{H\epsilon_i}, \quad \epsilon_0 = H$$

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} = 3\frac{\dot{\phi}^2/2}{\dot{\phi}^2/2 + V}$$

$$\epsilon_2 \equiv -\frac{\dot{\epsilon_1}}{H\epsilon_1} = -3\frac{\ddot{\phi}}{3H\dot{\phi}}$$

$$\epsilon_3 \equiv -\frac{\dot{\epsilon_2}}{H\epsilon_2} = 3(\epsilon_1 + \epsilon_2) - \epsilon_2^2 - \frac{V_{,\phi\phi}}{H^2}$$

These parameters quantify the degree of slow roll, depending on the shape of the potential. The background field is said to be slow-rolling if $\epsilon_1 \ll 1$, $|\epsilon_2| \ll 1$, ...

Inflation implies spatial flatness

Define

- critical density:

$$\rho_c(t) \;=\; 3H^2$$

- Density parameter:

$$\Omega(t) \;=\; \frac{\rho(t)}{\rho_c(t)}$$

Experimentally, $\Omega_0 \equiv \Omega(t_0) = \sum_i \Omega_i$ is inferred to be very close to one.

- To see how inflation drives Ω_0 towards one, rewrite the Friedmann equation including the spatial curvature term as:

$$\Omega - 1 = \frac{k}{a^2 H^2}$$

Inflation implies spatial flatness

- If the universe was always matter or radiation dominated in its past, then we get

$$\frac{k}{a^2 H^2} \propto \text{ growing function of time}$$

  $\Rightarrow$ if Ω_0 is so close to one today, it must be have been extraordinarily close to one in the past.

- If there was a period of $\ddot{a} > 0$, then during that period

$$\frac{k}{a^2 H^2} \propto \text{ decreasing function of time}$$

  So even if $\Omega(t)$ started out with a value far from one, inflation would drive it very quickly towards one. If this period was long enough, the subsequent radiation and matter domination epochs will not make it significantly different from one.

Thus, inflation implies

$$\Omega_0 \simeq 1$$

Inflationary attractor

- Inflation can be predictive only if the field evolution at late time is independent of the initial conditions, ϕ_i and $\dot{\phi}_i$. In other words, it must exhibit attractor behaviour.

- This means that in the phase space $(\phi, \dot{\phi})$, there must exist an attractor solution or trajectory, to which all solutions approach quickly. This attractor solution must contain an inflationary patch.

- The slow-roll solution generically gives a good approximation of the attractor trajectory.

Which is 'THE model' of inflation?

- What fundamental theory does ϕ belong to?

  - Extension of the Standard Model?
  - Low energy effective theory from some string theory compactification ?

- Alternative to inflation?

Inflaton perturbations and their nature

Nature of inflaton perturbations

$$\delta\phi \Longleftrightarrow \delta g_{\mu\nu}$$

Generic inflaton perturbation will be such that

$$
\begin{aligned}
\delta T_{ii} \equiv \delta p &= \frac{\partial p}{\partial \rho}\,\delta\rho + \delta p_{NA} \\
&= c_A^2\,\delta\rho + \delta p_{NA} \\
&= \text{adiabatic} + \text{nonadiabatic components}
\end{aligned}
$$

Perturbations induced by the inflaton

$$\delta\phi(\vec{x}, t) = \phi_0(t) + \delta\phi(\vec{x}, t)$$

▶ Metric fluctuations : perturb around the FRW (background) metric upto first order.

$$
\begin{aligned}
g_{\mu\nu} &\simeq g_{\mu\nu}^{(0)}(t) + \delta g_{\mu\nu}(\vec{x}, t), \\
\delta g_{\mu\nu}(\vec{x}, t) &\ll g_{\mu\nu}^{(0)}(t)
\end{aligned}
$$

▶ There are three types of perturbations depending on how they transform under local rotation of the spatial coordinates on hypersurfaces of constant time.

1. Scalar perturbations
2. Vector perturbations
3. Tensor perturbations

Counting the independent degrees of freedom

▶ The full perturbed metric:

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2A & \partial_i B + S_i \\ \partial_i B + S_i & (1 - 2\psi)\delta_{ij} + D_{ij}E - (\partial_i F_j + \partial_j F_i) + h_{ij} \end{pmatrix}$$

where,

$$D_{ij} = \partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2$$

▶ Four possible coordinate transformations eliminate 4 degrees of freedom. The remaining six include scalars, vectors and tensors.

Counting. . . *contd*

- At linear order, the three kinds of perturbations decouple. So they can be studied separately.
- Of the four gauge conditions possible, two will apply to scalars:
  gauge invariance under coordinate reparametrizations of time and space

$$
\begin{aligned}
t &\rightarrow t + \tilde{t} \\
x^i &\rightarrow x^i + \epsilon^i, \quad \epsilon^i = \partial_i \epsilon
\end{aligned}
$$

These remove two of the scalars.

- The remaining two gauge invariances under space transformations

$$x^i \to x^i + \epsilon^i, \quad \partial_i \epsilon^i = 0$$

  will act on the vector modes and remove two degrees of freedom.

- Thus, in the coupled system of inflaton perturbation and scalar metric perturbations, there are three independent variables - two metric scalar perturbations and one scalar field perturbation. The perturbed Klein Gordon equation and the perturbed Einstein equations will completely specify the dynamics of this coupled system.

- It turns out that, to linear order in perturbations, the Einstein equation give constraints which remove one more metric scalar variable.

From here on we will focus attention on the scalar variables only.

Perturbed Einstein equation

Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

.

Perturbed Einstein equation

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\delta g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\delta R$$

Since

$$\delta R = \delta g_{\mu\alpha}R_{\alpha\mu} + g_{\mu\alpha}\delta R_{\alpha\mu}$$

we need to only compute $\delta R_{\mu\nu}$:

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\mu \delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} - \delta\Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\delta\Gamma^\sigma_{\mu\alpha}$$

Perturbed Einstein equation

Components of $\delta G_{\mu\nu}$:

$$\delta G_{00} = 2\frac{a'}{a}\partial_i\partial^i B - 6\frac{a'}{a}\psi' + \frac{1}{2}\partial_k\partial^i D_{ki}E$$

$$\delta G_{0i} = -2\frac{a''}{a}\partial_i B + \left(\frac{a'}{a}\right)^2\partial_i B + 2\partial_i\psi' + \frac{1}{2}\partial_k D_{ki}E' + 2\frac{a'}{a}\partial_i A$$

$$\delta G_{ij} = \delta_{ij}\left[2\frac{a'}{a}A' + 4\frac{a'}{a}\psi'4\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A + 4\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi\right.$$

$$\left. 2\psi'' - \partial_k\partial^k\psi + 2\frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i A + \frac{1}{2}\partial_k\partial^m D_m^k E\right]$$

$$\partial_i\partial_j B' + \partial_i\partial_j\psi - \partial_i\partial_j A + \frac{a'}{a}D_{ij}E' - 2\frac{a''}{a}D_{ij}E + \left(\frac{a'}{a}\right)^2 D_{ij}E$$

$$+\frac{1}{2}D_{ij}E + \frac{1}{2}\partial_k\partial_i D_j^k E + \frac{1}{2}\partial^k\partial_i D_{jk}E - \frac{1}{2}\partial^k\partial_k D_{ij}E - 2\frac{a'}{a}B$$

Perturbed Energy momentum tensor

Unperturbed $T_{\mu\nu}$:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)$$

Perturbed $T_{\mu\nu}$:

$$
\begin{aligned}
\delta T_{\mu\nu} &= \partial_\mu\delta\phi\partial_\nu\phi + \partial_\mu\phi\partial_\nu\delta\phi - \delta g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)\\
&\quad g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + g^{\alpha\beta}\partial_\alpha\delta\phi\partial_\beta\phi - V,_\phi\,\delta\phi\right)
\end{aligned}
$$

Perturbed Energy momentum tensor

Components of $\delta T_{\mu\nu}$:

$$
\begin{aligned}
\delta T_{00} &= \delta\phi'\,\phi' + 2a^2 A\,V + a^2 V_{,\phi}\,\delta\phi \\
\delta T_{0i} &= \partial_i\delta\phi\,\phi' + \frac{1}{2}\partial_i B\phi'^2 - a^2\partial_i B V \\
\delta T_{ij} &= \delta^{ij}\left(\delta\phi'\,\phi' - A\phi'^2 - a^2 V_{,\phi}\,\delta\phi - \psi\phi'^2 + 2a^2\psi V\right) \\
&\qquad \frac{1}{2}D_{ij}E\phi'^2 - a^2 D_{ij}EV
\end{aligned}
$$

Perturbed Klein-Gordon equation

$$
\begin{aligned}
\delta\phi'' + 2\frac{a'}{a}\delta\phi' \quad &- \quad \partial_i\partial^i\delta\phi - A'\phi' - 3\psi'\phi' - \partial_i\partial^i B\phi' \\
&= \quad \delta\phi\frac{\partial^2 V}{\delta\phi^2}a^2 - 2A\frac{\partial B}{\partial\phi}
\end{aligned}
\tag{1}
$$

Gauge transformations

Let

$$
\begin{aligned}
\delta x^0 &= \xi^0(x^\mu) \\
\delta x^i &= \partial^i \beta(x^\mu) + v^i(x^\mu) : \quad \partial_i v^i = 0.
\end{aligned}
$$

▶ Transformation properties of the four metric perturbation scalars:

$$
\begin{aligned}
\tilde{A} &= A - \xi^{0\prime} - \frac{a'}{a}\xi^0 \\
\tilde{B} &= B + \xi^0 + \beta' \\
\tilde{\psi} &= \psi - \frac{1}{3}\nabla^2\beta + \frac{a'}{a}\xi^0 \\
\tilde{E} &= E + 2\beta
\end{aligned}
\tag{2}
$$

▶ The inflaton perturbation transforms as

$$
\tilde{\delta\phi} = \delta\phi - \phi'\xi^0
\tag{3}
$$

Gauge invariant quantities

Which dynamical variable best carries the physical information ?

- ▶ Gauge invariant Bardeen potentials:

$$
\begin{aligned}
\Phi &= -A + \frac{1}{a}\left[a\left(-B + \frac{E'}{2}\right)\right]' \\
\Psi &= -\psi - \frac{1}{6}\nabla^2 E + \frac{a'}{a}\left(B - \frac{E'}{2}\right)
\end{aligned}
$$

- ▶ Gauge invariant inflaton perturbation

$$
\delta\phi^{(GI)} = -\delta\phi - \phi'\left(-B + \frac{E'}{2}\right) \tag{4}
$$

- ▶ Gauge invariant energy-density perturbation

$$
\delta\rho^{(GI)} = -\delta\rho - \rho'\left(-B + \frac{E'}{2}\right) \tag{5}
$$

Gauge invariant perturbed Einstein equation

- By defining gauge invariant components of $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$ we can write Einstein equation in a gauge invariant way.

$$
\begin{aligned}
\delta G_{00}^{(GI)} &= \delta G_{00} + (\delta G_{00})' \left(\frac{E'}{2} - B \right) \\
\delta G_{0i}^{(GI)} &= \delta G_{0i} + (\delta G_{0i})' \left(\frac{E'}{2} - B \right) \\
\delta G_{ij}^{(GI)} &= \delta G_{ij} + (\delta G_{ij})' \left(\frac{E'}{2} - B \right) \\
\delta T_{00}^{(GI)} &= \delta T_{00} + (\delta T_{00})' \left(\frac{E'}{2} - B \right) \\
\delta T_{0i}^{(GI)} &= \delta T_{0i} + (\delta T_{0i})' \left(\frac{E'}{2} - B \right) \\
\delta T_{ij}^{(GI)} &= \delta T_{ij} + (\delta T_{ij})' \left(\frac{E'}{2} - B \right)
\end{aligned}
$$

Gauge invariant perturbed Einstein equation

- We can write the Einstein's equations in terms of the Bardeen potentials, Φ and Ψ.

- For $i \neq j$ we get

$$\Phi = \Psi$$

- We finally get the following equation for Φ:

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi}\right)\Phi' - \nabla^2\Phi + 2\left(2\mathcal{H}' + \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi = 0$$

Mukhanov-Sasaki variable

- Define the Mukhanov-Sasaki variable u as:

$$u \;\equiv\; a\,\delta\phi^{(GI)} + z\,\Phi$$

where

$$z \;\equiv\; \frac{a}{H}\sqrt{\rho + p} \;=\; a\,\frac{\dot{\phi}}{H}$$

u is gauge invariant by consruction.

- The equation of motion for u then becomes

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0.$$

For the sake of completeness, once u is calculated, we can calculate Φ via the equations:

$$\nabla^2\Phi = \frac{\mathcal{H}}{2a^2}(zu' - z'u)$$

Comoving curvature perturbation

- Define the comoving curvature perturbation as:

$$\mathcal{R} \equiv -\frac{H}{\phi'}\delta\phi^{(GI)} - \Phi = -\frac{u}{z}$$

- Equation of motion of $\mathcal{R}$:

$$\mathcal{R}'' - \nabla^2\mathcal{R} + 2\frac{z'}{z}\mathcal{R}' = 0$$

- $\mathcal{R}$ is gauge invariant by construction.

Physical meaning of $\mathcal{R}$: On comoving hypersurfaces where $\delta\phi = 0$, $\mathcal{R} = \psi$ and so it gives the intrinsic spatial curvature via the Poisson equation:

$$R^{(3)} = \frac{4}{a^2}\nabla^2\psi$$

Intrinsic Entropy perturbation

If the pressure does not change adiabatically, then it will not only be a function of ρ.

- Let
$$p \equiv p(\rho, S)$$

  Then
$$\delta p = c_A^2 \, \delta\rho + \frac{\dot{p}}{H} \mathcal{S}$$

- Equivalent to defining
$$\mathcal{S} \equiv H \left(\frac{\delta\rho}{\dot{\rho}} - \frac{\delta p}{\dot{p}} \right)$$

- It is gauge invariant by definition.

$\mathcal{S}$ can be shown to be

$$\mathcal{S} = \frac{2V_{,\phi}}{3\dot{\phi}(3H\dot{\phi} + 2V_{,\phi})} \left[\dot{\phi} \left(\dot{\delta\phi} - \dot{\phi}\Phi \right) - \ddot{\phi}\delta\phi \right]$$

Second order action in Mukhanov-Sasaki variable and quantization

To quadratic order in the perturbations, the inflaton action can be written in terms of the Mukhanov-Sasaki variable as:

$$S = \int d\eta d^3x \, \frac{1}{2} \left(u'^2 - c_s^2 \, u \, \nabla^2 u + \frac{z''}{z} u \right)$$

where $c_s^2 = 1$.

▶ It is equivalent to a free scalar field with time dependent mass, given by z''/z, in Minkowski space.

▶ The vacuum is chosen to be the Bunch Davies vacuum, in the limit $aH/k \rightarrow 0$

$$u_k \longrightarrow \frac{1}{\sqrt{2k}} e^{-ik\eta} \tag{6}$$

▶ The quantum operator $u(\vec{x}, \eta)$ can be expressed as:

$$u(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(u_k \, a_k e^{i\vec{k}.\vec{x}} + u_k^*(\eta) \, a_k^\dagger e^{-i\vec{k}.\vec{x}} \right)$$

Solution for u, $\mathcal{R}$ and $\mathcal{S}$ under slow roll on super-Hubble scales

Solutions for perturbations

To solve the classical equations of motion for $u, \mathcal{R}, \mathcal{S}$ we consider the equations for the Fourier modes

▶ Equation of motion for Fourier modes of u:

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0.$$

▶ Equation of motion of Fourier modes of $\mathcal{R}$:

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2 \mathcal{R}_k = 0$$

It is useful to have the expressions for z''/z and z'/z expressed in terms of slow roll parameters:

$$
\begin{aligned}
\frac{z''}{z} &= 2a^2 H^2 \left(1 + \epsilon_1 - \frac{3}{2}\epsilon_2 - 2\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_1^2 + \frac{1}{2}\epsilon_3\right) \\
\frac{z'}{z} &= aH\left(1 + \epsilon_1 - \epsilon_2\right)
\end{aligned}
$$

Solutions for perturbations

W will need solutions in the following two limits:

- Sub-Hubble scales:

$$\frac{k}{aH} \gg 1$$

- Super-Hubble scales:

$$\frac{k}{aH} \ll 1$$

Using the expression for z''/z in slow roll parameters, the equation for u_k becomes:

$$u_k'' + a^2 H^2 \left(\frac{k^2}{a^2 H^2} - 2(1 + \epsilon_1 - \frac{3}{2}\epsilon_2 - 2\epsilon_1\epsilon_2 + \frac{1}{2}\epsilon_1^2 + \frac{1}{2}\epsilon_3) \right) u_k = 0.$$

Solutions for perturbations

First, let us see that $\mathcal{R}_k$ is approximately zero on super-Hubbles scales if the backgound evolution is slow-roll.

In the two scale limits u_K equation becomes :

- ▶ On sub-Hubble scales

$$u_k'' + k^2 u_k \simeq 0$$

- ▶ On super-Hubble scales

$$u_k'' - \frac{z''}{z} u_k \simeq 0$$

- ▶ On super-Hubble scales we have: $u \propto z$.
- ▶ This implies:

$$\dot{\mathcal{R}} \simeq 0$$

Solutions for perturbations

1. Growing mode:
$$\mathcal{R}_1(\eta) \simeq \text{const}$$

2. Decaying mode: can be obtained from $\mathcal{R}_1$ by using the Wronskian

$$\mathcal{R}_2(\eta) \; \propto \; \mathcal{R}_1(\eta) \int_{\eta_*}^{\eta} \; \frac{d\eta'}{z^2(\eta') \, \mathcal{R}_1^2(\eta')}$$

$\mathcal{R}$ can then be written as:

$$\mathcal{R}_k(\eta) = c_1 \mathcal{R}_1(\eta) + c_2 \mathcal{R}_2(\eta)$$

Under slow roll, $\mathcal{R}_2(\eta) \ll \mathcal{R}_2(\eta_k)$ for $\eta \gg \eta_k$. and so

$$\mathcal{R}_k \simeq \mathcal{R}_1$$

Solutions for perturbations

▶ Relation between Curvature and Intrinsic Entropy perturbation

Using the expression for $\mathcal{S}$, the equation for $\mathcal{R}_k$ can be rewritten as

$$\frac{\mathcal{R}'_k}{aH} = \frac{3}{2}\frac{3 - 2\epsilon_2}{3 - \epsilon_2}\mathcal{S}$$

$$\begin{aligned}\frac{\mathcal{S}'_k}{aH} &= \left(\frac{3(\epsilon_1\epsilon_2 - \epsilon_3)}{(3 - 2\epsilon_2)(3 - \epsilon_2)} - 3 - \epsilon_1 + 2\epsilon_2\right)\mathcal{S}\\ &\quad - \frac{2}{3}\frac{(3 - \epsilon_2)}{(3 - 2\epsilon_2)}\frac{k^2}{a^2H^2}\mathcal{R}\end{aligned}$$

Assuming $\epsilon_1 \ll 1$ and $V_{,\phi\phi} \ll H^2$,

$$\frac{\mathcal{S}'_k}{aH} = \left(2\epsilon_2 - 3 + \frac{3\epsilon_2}{2\epsilon_2 - 3}\right)\mathcal{S}$$

Solutions for perturbations

- Intrinsic Entropy perturbations behave as

$$\mathcal{S} \propto e^{-2N}$$

- which also implies

$$\dot{\mathcal{R}} \to 0$$

when the mode goes out of the horizon.

Solutions for perturbations

To Solve the u_k equation :

First we need to know how a behaves as a function of conformal time η. FRom

$$\eta = \int \frac{dt}{a}$$

we get

For nearly de-Sitter expansion, $a(\eta)$ goes as

$$a(\eta) \simeq -\frac{1}{H\eta} + \mathcal{O}(\text{slow roll parameters})$$

Therefore,

$$\frac{k}{aH} \ll 1 \quad \Rightarrow \quad -k\eta \ll 1$$

and similarly for the other limit. Then we can write

$$\frac{z''}{z} \simeq \frac{1}{\eta^2}(2 + \mathcal{O}(\text{slow roll parameters})$$

Solutions for perturbations

Rewrite the u_k equation as:

$$u_k'' + \left(k^2 - \frac{1}{\eta^2} \left(\nu^2 - \frac{1}{4} \right) \right) u_k = 0$$

where

$$\nu = \frac{3}{2} + \mathcal{O}(\text{slow roll parameters})$$

For ν real, as it is here, the general solution of this equation is:

$$u_k(\eta) = \sqrt{-\eta} \, \left(c_1(k) \, H_\nu^{(1)}(-k\eta) + c_2(k) \, H_\nu^{(2)}(-k\eta) \right)$$

where $H_\nu^{(1)}(-k\eta)$ and $H_\nu^{(2)}(-k\eta)$ are Hankel functions of the first and second kind. The coeeficents $c_1(k)$ and $c_2(k)$ are independent of time.

Solutions for perturbations

▶ In the limit $-k\eta \gg 1$, we have:

$$H_\nu^{(1)}(-k\eta) \simeq \sqrt{\frac{2}{-k\,\eta\,\pi}}\,e^{i\left(-k\eta-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}$$

$$H_\nu^{(2)}(-k\eta) \simeq \sqrt{\frac{2}{-k\,\eta\,\pi}}\,e^{-i\left(-k\eta-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}$$

▶ Next we demand that in the limit $k/aH \gg 1$ or $-k\eta \gg 1$ we get plane wave solution:

$$u_k \to \frac{1}{\sqrt{2k}}e^{-ik\eta}$$

▶ This implies that we must have:

$$c_2(k) = 0$$
$$c_1(k) = \frac{\sqrt{\pi}}{2}e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}$$

Since c_1 actually depends on time through the slow roll parameters in ν, this solution is valied under the

Solutions for perturbations

Thus the solution for u_k becomes:

$$u_k = \frac{\sqrt{\pi}}{2} \, e^{i(\nu + \frac{1}{2})\frac{\pi}{2}} \, \sqrt{-\eta} \, H_\nu^{(1)}(-k\eta)$$

In the limit $k/aH \ll 1$ or $-k\eta \ll 1$, $H_\nu^{(-1)}$ behaves as

$$H_\nu^{(-1)} \simeq \sqrt{\frac{2}{\pi}} \, e^{-\frac{i\pi}{2}} \, 2^{\nu - \frac{3}{2}} \, \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{(-k\eta)^\nu}$$

we get

$$
\begin{aligned}
u_k &\simeq e^{i(\nu - \frac{1}{2})\frac{\pi}{2}} \, 2^{\nu - \frac{3}{2}} \, \frac{\Gamma(\nu)}{\Gamma(3/2)} \, \frac{1}{\sqrt{2k}} \, (-k\eta)^{\frac{1}{2} - \nu} \\
&= \left[2^{\nu - \frac{3}{2}} \, \frac{\Gamma(\nu)}{\Gamma(3/2)} \right] \, \frac{1}{\sqrt{2k}} \, \left(\frac{k}{aH} \right)^{\frac{1}{2} - \nu}
\end{aligned}
$$

Solutions for perturbations

We can now obtain $\mathcal{R}_k$ as:

$$
\begin{aligned}
|\mathcal{R}_k| &= \left|\frac{u_k}{z}\right| \simeq \left[2^{\nu-\frac{3}{2}}\,\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]\,\frac{H^2}{\dot{\phi}}\frac{1}{aH\sqrt{2k}}\,\left(\frac{k}{aH}\right)^{-1+(n_s-1)/2} \\
&= \left[2^{\nu-\frac{3}{2}}\,\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]\,\frac{H^2}{\dot{\phi}}\,\frac{1}{\sqrt{2k^3}}\,\left(\frac{k}{aH}\right)^{(n_s-1)/2}
\end{aligned}
$$

where

$$
n_s = 1 - 2\epsilon_1 - \epsilon_2
$$

Statistical properties

Two-point correlation function: power spectrum

► For a generic operator, A, the power spectrum $\mathcal{P}(k)$ is defined from the two-point function as:

$$\langle 0|A^*_{\vec{k}_1} A_{\vec{k}_2}|0\rangle \equiv \delta(\vec{k}_1 - \vec{k}_2)\frac{2\pi^2}{k^3}\mathcal{P}(k)$$

where $\mathcal{P}(k)$ is the power spectrum.

► The scalar power spectrum for $\mathcal{R}$ is :

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\,\pi^2}\,|\mathcal{R}_{\mathrm{k}}|^2$$

where $\mathcal{R}_k$ here is the solution of the classical equation of motion.

Power spectrum of $\mathcal{R}$

▶ The expression for $P_{\mathcal{R}}$ is :

$$P_{\mathcal{R}} \simeq \frac{1}{4\pi} \left[2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \right]^2 \frac{H^4}{\dot{\phi}^2} \left(\frac{k}{aH} \right)^{n_s-1}$$

Hence the power spectrum can be written as

$$P_{\mathcal{R}}(k) = A \left(\frac{k}{k_0} \right)^{n_s-1}$$

where $k_0 = $ pivot scale, $A = $ amplitude, and n_s is the spectral index:

$$n_s = \frac{d \ln P_{\mathcal{R}}}{d \ln k/k_0}$$

Power spectrum of $\mathcal{R}$

- Thus, $P_{\mathcal{R}}(k)$ is *featureless* and completely specified by two numbers, A and n_s.
- The scale dependence (dependence on k) is very weak, since under slow roll n_s is close to one.

Deviations from slow roll

- When slow roll is violated, $\mathcal{R}_k$ can have significant evolution on super-Hubble scales.
- The evolution/change can be quantified using long wavelength approximation.

Approximate formula for the change of $\mathcal{R}_k$

Leach, Sasaki, Wands and Malik (2001), Jain, PC, Sriramkumar (2007)

▶ The long wavelength approximation assumes a perturbative expansion for the growing mode $\mathcal{R}_1$ in powers of the wavevector k:

$$\mathcal{R}_1(\eta) = \sum_{n=0}^{\infty} R_{1(n)}(\eta) k^{2n}.$$

such that $\mathcal{R}_{1(0)}$ is the asymptotic constant solution in the limit $k \to 0$.

Approximate formula for the change of $\mathcal{R}_k$

- Let $\eta_* = $ end of inflation
  $\eta_k = $ soon after Hubble exit.

- Let R_K at these two times be related as:

$$\mathcal{R}_k(\eta_*) = \alpha_k \mathcal{R}_k(\eta_k).$$

- Then to order k^2, α_k is given by:

$$\alpha_k = 1 + D_k(\eta_k) - F_k(\eta_k)$$

with

$$D_k(\eta) \simeq \mathcal{H}_k \int\limits_{\eta}^{\eta_*} d\eta_1 \, \frac{z^2(\eta_k)}{z^2(\eta_1)}$$

and

$$F_k(\eta) \simeq k^2 \int\limits_{\eta}^{\eta_*} \frac{d\eta_1}{z^2(\eta_1)} \int\limits_{\eta_k}^{\eta_1} d\eta_2 \, \left[c_{\mathrm{s}}^2(\eta_2) \, z^2(\eta_2) \right]$$

$\mathcal{H}_k = $ conformal Hubble parameter at η_k.

Entropy perturbations as source for $\mathcal{R}_k$

Leach and Liddle (2001), Jain, PC, Sriramkumar (2007)

During fast roll, ϵ_2 is larger than one and so

$$\mathcal{S} \propto e^{cN}, \qquad c > 1$$

• This growth sources $\mathcal{R}_k$ and makes it evolve. The effect is localized to those modes that exit Hubble horizon around the period of slow roll violation. The longer a mode k has been outside the horizon, the lesser is the effect.

• The detailed effects are model dependent.

Non-gaussianity

Seery and Lidsey (2005)

Consider a general action of the form

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R + P(X, T) \right)$$

where $X = -g^{\mu\nu} \partial_\mu T \partial_\nu T$.

$$\dot{X} (P_{,X} + 2XP_{,XX}) + 2\sqrt{3} (2XP_{,X} - P)^{1/2} XP_{,X}$$
$$= X^{1/2} (P_{,T} - 2XP_{,XT}),$$

and a constraint equation, which is the Friedmann equation,

$$H^2 = \frac{1}{3} (2XP_{,X} - P).$$

The sound speed in T is given by

$$c_s^2 = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}.$$

Define the slow roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{XP_X}{H^2} \;\; = \;\; -\frac{\dot{T}}{H^2}\frac{\partial H}{\partial T} - \frac{\dot{X}}{H^2}\frac{\partial H}{\partial X}$$
$$= \;\; \epsilon_T + \epsilon_X.$$

$$\eta = \frac{\dot{\epsilon}}{\epsilon H}.$$

It is also useful to define the parameters u and s as

$$u \;\; = \;\; 1 - \frac{1}{c_s^2} = -\frac{2XP_{XX}}{P_X},$$
$$s \;\; = \;\; \frac{1}{H}\frac{\dot{c}_s}{c_s}.$$

Three-point function of Curvature perturbation

Starting with the above general action, follow the steps below:

1. Expand the action order by order in the curvature perturbation $\mathcal{R}$.

$$S = S_0 + S_1 + S_2 + S_3 + \ldots$$

2. The zeroth order terms, S_0, will give the background evolution.

3. First order terms are zero, S_1.

4. The second order action, S_2, describes a free field.

5. Determine the terms in the third order action S_3. It can be written as an expansion in the slow roll parameters - $\epsilon_1, \epsilon_2, \varepsilon_X, u$ and c_s.

Three-point function of Curvature perturbation

$$\langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\mathcal{R}(\vec{k}_3)\rangle = (2\pi)^3 \delta\left((\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\frac{H^4}{2^4\epsilon_1^2}\frac{1}{\Pi_i}k_i^3\mathcal{A}\right)$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{4}{K}(u+\epsilon_1)\sum_{i>j}k_i^2k_j^2 - \frac{4}{K^3}\left(u+\frac{\epsilon_1}{\varepsilon_X}\frac{s}{3}\right)k_1^2k_2^2k_3^2 \\
&\quad -\frac{2u}{K^2}\sum_{i\neq j}k_i^2k_j^3 + \frac{1}{2}(\epsilon_2 - u - \epsilon_1)\sum_i k_i^3 + \frac{\epsilon_1}{2}\sum_{i\neq j}k_ik_j^2
\end{aligned}$$

where $K = k_1 + k_2 + k_3$.

Non-gaussianity parameter : f_{NL}

Let

$$f_{NL} = -\frac{5}{6}\frac{\mathcal{A}}{\sum_i k_i^3}$$

For the case $k_1 = k_2 = k_3$, the expression for f_{NL} is given by

$$f_{NL} \simeq -0.28\,u + 0.02\,\frac{\epsilon_1}{\varepsilon_X}\,s - 1.53\,\epsilon_1 - 0.42\,\epsilon_2.$$

For canonical scalar field inflation, $c_s = 1$, $u = 0$ and $s = 0$.
Therefore,

$$f_{NL} \simeq -1.53\,\epsilon_1 - 0.42\,\epsilon_2$$

and so

$$f_{NL} \ll 1$$

.

Summary of lectures 1 and 2

Studied inflation and properties of perturbations arising from a single scalar field minimally coupled to gravity. We obtained the following general predictions:

- $\Omega_0 = 0$.
- Inflaton perturbations are adiabatic and 'conserved' on super-Hubble scales if the field dynamics is slow roll always.
- The power spectrum of the quantized perturbations is 'featureless' and nearly scale invariant. It is specified by two numbers: the amplitude A and the spectral index n_s.
- Perturbations behave as free fields to linear order and hence are gaussian. The non-gaussianity predicted is small.
- If slow roll is violated for some time during inflation, features arise in the power spectrum. These features are associated with relatively large non-gaussianity, at the scales associated with the features.

Classification of inflation models

Based on number of fields driving inflation:

1. single field inflation
2. multi-field inflation

Based on initial field value:

1. Large field inflation
2. Small field inflation

Based on the type of scalar field:

1. Canonical scalar field inflation
2. Dirac Born Infeld field inflation
   2.1 Tachyonic inflation: $L = -V(T)\sqrt{1 + \partial_\mu \partial^\mu T}$
   2.2 'DBI' inflation:
       $L = -f(T)\sqrt{1 + f^{-1}(T)\partial_\mu \partial^\mu T} + f(T) - V(T)$

Hybrid inflation, curvaton scenario, . . .

Some issues not mentioned

- Reheating
- Does reheating details affect super-Hubbles perturbations?
- Trans-Planckian effects
- Quantum to classical transition
- All quantum field theoretic calculations are at tree level. Loop corrections are expected to be small.

Counting. . . *contd*

- Helmholtz' theorem: any three vector u^i can always be written as sum of a curl free part and a divergence free part:

$$u^i = \partial_i v + v^i \quad \text{with} \quad \nabla \mathbf{v} = \mathbf{0} \tag{7}$$

  The divergence free condition implies the vector actually has two independent degrees of freedom.

- A general traceless tensor Π_{ij} can be decomposed as

$$
\begin{aligned}
\Pi_{ij} &= \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T \\
\Pi_{ij}^S &= \left(-\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij} \right) \Pi \\
\Pi_{ij}^V &= -\frac{i}{2k} \left(k_i \Pi_j + k_j \Pi_i \right), \quad \text{with} \quad k_i \Pi_i = 0, \\
k_i \Pi_{ij}^T &= 0.
\end{aligned}
\tag{8}
$$

  So, the number of independent tensor degrees of freedom are two.

- Thus we are left with two independent scalar degrees of freedom.

Gauge transformations

How do the perturbed parts of various quantities transform under a general coordinate transformation?

Let $Q(x^\mu)$ be some generic quantity which may be a scalar, vector or tensor, which is perturbed as

$$Q(x^\mu) = Q_0(t) + \delta Q(x) \tag{9}$$

Under the change of coordinates:

$$x^{\mu'} = x^\mu + \delta x^\mu \tag{10}$$

Let

$$
\begin{aligned}
\delta x^0 &= \xi^0(x^\mu) \\
\delta x^i &= \delta\beta(x^\mu) + v^i(x^\mu): \quad \partial_i v^i = 0.
\end{aligned}
$$

δQ transforms as

$$\delta Q'(x^{\mu'}) = \delta Q + \mathcal{L}_{\delta\xi} Q_0 \tag{11}$$