Anomalies of discrete symmetries

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（威） （荒木）

- T.A, *Prog. Theor. Phys.* *117* (2007).
- T.A, T.Kobayashi, J.Kubo, S.R.Sanchez, M.Ratz, and P.K.S.Vaudrevange, *arXiv:0805.0207 (hep-th)*.

Who is Takeshi Araki?

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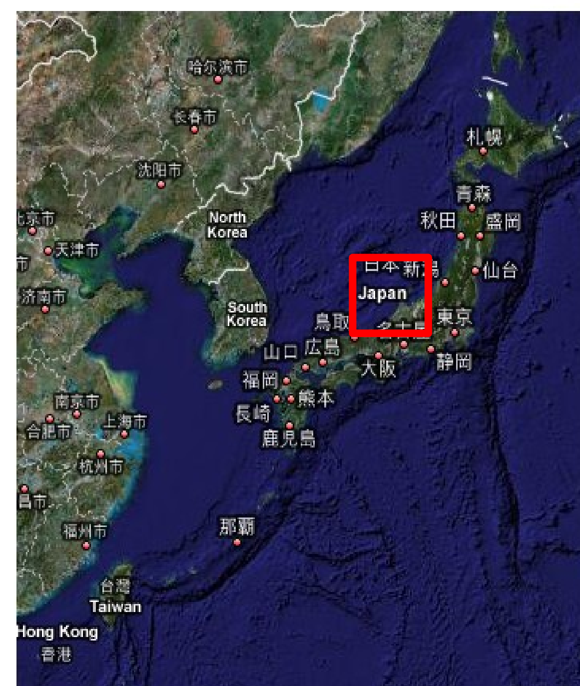
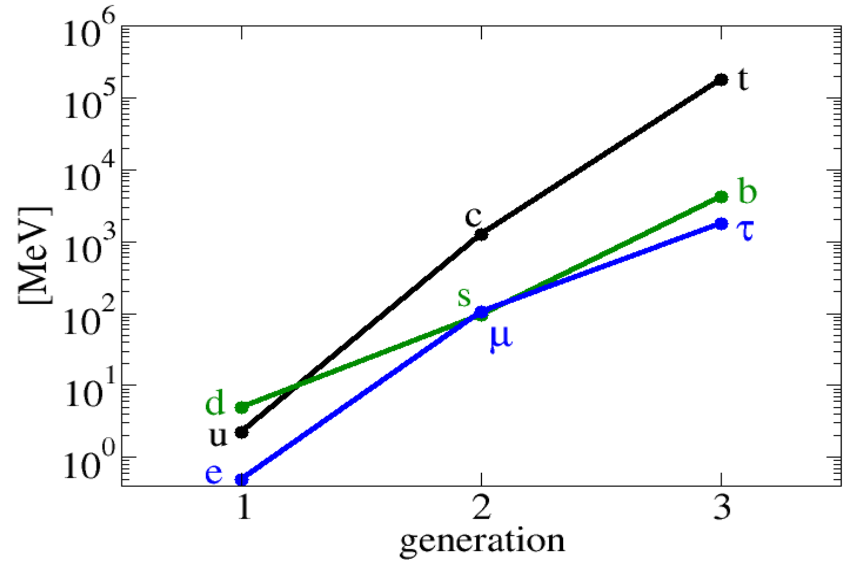


Discrete flavor symmetries

Recently, a lot of models with a flavor symmetry based on a finite group have been proposed ($S_3, A_4, D_N, Q_N \cdots$), because it is possible....

- to reduce the number of free parameters in the Yukawa sector.
- to realize hierarchical structure of fermion masses.
- to realize bi-maximal structure of MNS matrix.



V_{CKM} small mixing **VS** bi-large mixing V_{MNS}

quark lepton

Motivation

There are too many models!

Most of the models are meaningless because a flavor symmetry is only one (if there exist it).

But we have no criterion to distinguish them.

I thought it is important to consider a new criterion.

Anomalies of discrete symmetries.

That is, I want to consider a thing such as gauge anomalies.

$$\mathcal{A}_{[U(1)_Y]^3} = \sum_f \left[(Y_L^f)^3 - (Y_R^f)^3 \right] = 0$$

$$\mathcal{A}_{[SU(2)_L]^2-U(1)_Y} = \sum_f Y_L^f \ell_2(\mathsf{r}^f) = 0$$

$$\mathcal{A}_{[SU(3)_C]^2-U(1)_Y} = \sum_f Y_L^f \ell_3(\mathsf{r}^f) = 0$$

$$\mathcal{A}_{[U(1)_Y]^2-D} =$$

$$\mathcal{A}_{[SU(2)_L]^2-D} =$$

$$\mathcal{A}_{[SU(3)_C]^2-D} =$$

?

Motivation

To this end, there are two problems.

1, How do I define anomalies
for discrete symmetries?

2, Why are discrete anomalies forbidden?

I would like to talk about these topics in this presentation.

Fujikawa's method

[Fujikawa, PRL 42(1979)]

Let us consider a continuous chiral rotation, $\psi \to e^{i\alpha(x)\gamma_5}\psi$, in Euclidean space-time.

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \;\; \mathcal{D}A_\mu \;\exp\left\{\int d^4x \;\; \mathcal{L}(\psi, A)\right\}$$

$$\mathcal{L}' = \mathcal{L} + \alpha(x)\partial_\mu j_5^\mu$$

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to J^{-1}\mathcal{D}\bar{\psi}\mathcal{D}\psi$$

$$J^{-1} = \left\{\det \int d^4x \; \varphi_n^\dagger(x)\left[e^{i\alpha(x)\gamma_5}\right]\varphi_m(x)\right\}^{-2}$$

$$e^{i\alpha(x)\gamma_5} \simeq 1 + i\alpha(x)\gamma_5$$

$$\simeq \exp\left\{-2\sum_n^\infty \int d^4x \; \varphi_{n,i}^\dagger(x)\left[i\alpha(x)\gamma_5\right]_{ij}\varphi_{n,j}(x)\right\}$$

$$= \exp\left\{-i\int d^4x \; \frac{\alpha(x)}{16\pi^2}\mathrm{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(A)F_{\rho\sigma}(A)\right]\right\}$$

$$\partial_\mu j_5^\mu \neq 0$$

Hence the current does not conserve at the quantum level.

Discrete anomalies

I want to consider the same calculation for discrete ones. Let us consider a discrete chiral rotation, $\psi \to e^{i\alpha\gamma_5}\psi$, in Euclidean space-time. For instance $\alpha = (2\pi/N)q$.

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \quad \mathcal{D}A_\mu \ \exp\left\{\int d^4x \quad \mathcal{L}(\psi, A)\quad\right\}$$

$$\mathcal{L}' = \mathcal{L}$$

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \to J^{-1}\mathcal{D}\bar{\psi}\mathcal{D}\psi$$

$$J^{-1} = \left\{\det \int d^4x \ \varphi_n^\dagger(x)\left[e^{i\alpha\gamma_5}\right]\varphi_m(x)\right\}^{-2}$$

?

In this case, we can not use an infinitesimal transformation because α is a discrete parameter.

Discrete anomalies

We can expand the Jacobian as follows.

$$J^{-1} = \left\{ \det \int d^4x \; \varphi_n^\dagger(x) \left[e^{i\alpha\gamma_5} \right] \varphi_m(x) \right\}^{-2} \equiv \{\det C_{nm}\}^{-2}$$

Definition

$$D\varphi_n(x) = \lambda_n \varphi_n(x)$$

$$\psi_i(x) = \sum_n a_n \varphi_{n,i}(x)$$

$$\sum_n^\infty \varphi_{n,i}(x)\varphi_{n,j}^\dagger(y) = \delta_{ij}\delta^4(x-y)$$

$$\int d^4x \; \varphi_{n,i}^\dagger(x)\varphi_{m,i}(x) = \delta_{nm}$$

$$C_{nm} = \int d^4x \; \varphi_n^\dagger(x) \left[e^{i\alpha\gamma_5} \right] \varphi_m(x) = \delta_{nm} + \int d^4x \; \varphi_{n,i}^\dagger(x) \left[i\alpha\gamma_5 \right]_{ij} \varphi_{m,j}(x)$$

$$+ \int d^4x \; \varphi_{n,i}^\dagger(x) \frac{1}{2!} \left[i\alpha\gamma_5 \right]_{ij}^2 \varphi_{m,j}(x)$$

$$+ \int d^4x \; \varphi_{n,i}^\dagger(x) \frac{1}{3!} \left[i\alpha\gamma_5 \right]_{ij}^3 \varphi_{m,j}(x) + \cdots$$

$$= \delta_{nm} + \tilde{C}_{nm} + \frac{1}{2!}\tilde{C}_{nm}^2 + \frac{1}{3!}\tilde{C}_{nm}^3 + \cdots$$

$$\begin{aligned} \int d^4x \; \varphi_{n,i}^\dagger(x) \left[i\alpha\gamma_5 \right]_{ij}^2 \varphi_{m,j}(x) &= \int d^4x \int d^4y \; \varphi_{n,i}^\dagger(x) \left[i\alpha\gamma_5 \right]_{ij} \delta_{jk}\delta^4(x-y) \left[i\alpha\gamma_5 \right]_{kl} \varphi_{m,l}(y) \\ &= \int d^4x \int d^4y \; \varphi_{n,i}^\dagger(x) \left[i\alpha\gamma_5 \right]_{ij} \varphi_{p,j}(x)\varphi_{p,k}^\dagger(y) \left[i\alpha\gamma_5 \right]_{kl} \varphi_{m,l}(y) \\ &= \tilde{C}_{np}\tilde{C}_{pm} = \tilde{C}_{nm}^2 \end{aligned}$$

Then we use $\det A = \exp\{\mathrm{Tr}\ln A\}$ to obtain

$$\begin{aligned} J^{-1} &= \det \left\{ \int d^4x \; \varphi_{n,i}^\dagger(x) \left(e^{i\alpha\gamma_5} \right)_{ij} \varphi_{m,j}(x) \right\}^{-2} \\ &= \exp\left\{ -2 \sum_n^\infty \int d^4x \; \varphi_{n,i}^\dagger(x) \left[i\alpha\gamma_5 \right]_{ij} \varphi_{n,j}(x) \right\} \\ &= \exp\left\{ -i \int d^4x \; \frac{\alpha}{16\pi^2} \mathrm{Tr}\left[\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right] \right\}. \end{aligned}$$

Remaining calculation is the same as that of conventional one.

Anomalies of non-abelian discrete symmetries

Let us consider non-abelian discrete flavor transformations.

$$\psi_{L,\alpha} \to U_{\alpha\beta}\psi_{L,\beta} \equiv (e^{iX})_{\alpha\beta}\psi_{L,\beta}$$
$$\psi_{R,\alpha} \to V_{\alpha\beta}\psi_{R,\beta} \equiv (e^{iY})_{\alpha\beta}\psi_{R,\beta}$$

(α, β : generations)

These are unitary transformations ($UU^\dagger = VV^\dagger = 1$).

$$\det(U) = \det(e^{iX}) = e^{i\eta}$$
$$\det(V) = \det(e^{iY}) = e^{i\xi}$$

$$\eta, \xi \propto \frac{2\pi}{N}$$

Let us calculate the Jacobian for these transformations.

$$J^{-1} = \exp\left\{\frac{\eta i}{32\pi^2}\int d^4x \ \mathrm{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(L)F_{\rho\sigma}(L)\right]\right\}$$
$$\times \exp\left\{-\frac{\xi i}{32\pi^2}\int d^4x \ \mathrm{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(R)F_{\rho\sigma}(R)\right]\right\}$$

Here, η and ξ correspond to the abelian parts of U and V .

Therefore, we only have to take into account its abelian parts.

Stringy originated discrete symmetries

We want to consider the situation that discrete flavor anomalies are forbidden. ⇒ But there is no reason....

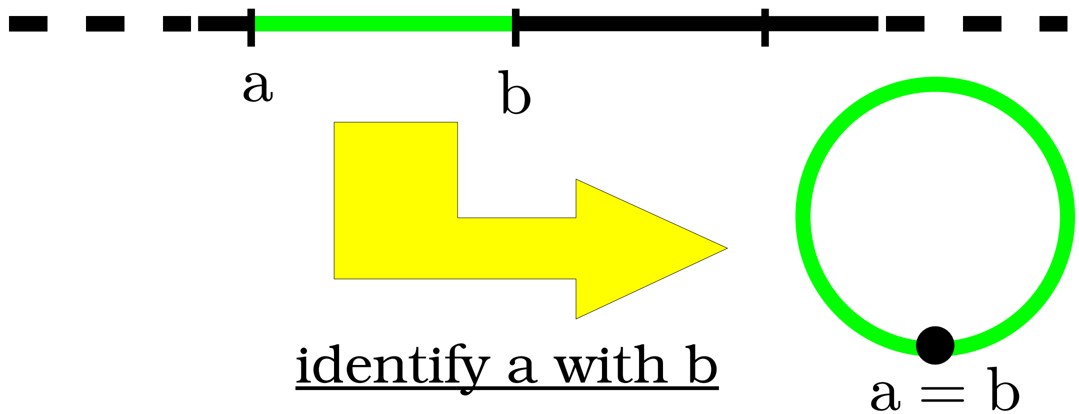
We assumed that discrete flavor symmetries originate from string theory.

⇒
- In fact, we can derive non-abelian discrete flavor symmetries from heterotic orbifold models. [T.Kobayashi, *et al.* NPB768(2007)]
- Such discrete symmetries reflect geometrical symmetries of internal space.
- The geometrical operations are embedded into the gauge group.
- There might be some relation between discrete anomalies and that of anomalous U(1) gauge symmetries.

We expected that we might get some constraint for discrete anomalies from anomaly freedom of anomalous U(1) if there exist such a relation.

S^1/Z_2 [T.Kobayashi, *et al.* *NPB*768(2007)]

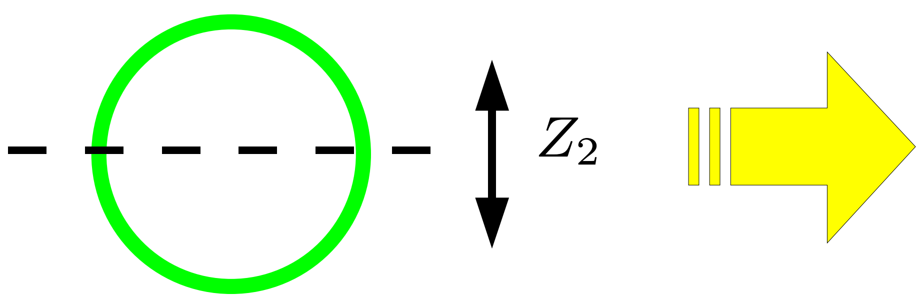
Divide a 1-dimensional space by a 1-dimensional lattice.

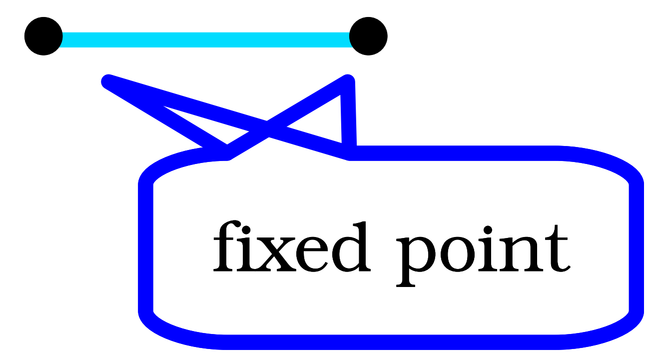


S^1 torus

Identify the points which are related by a reflection.

S^1/Z_2 orbifold



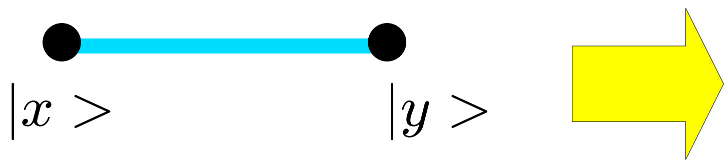


The points of both ends are invariant under Z_2- twist.

There are brane matter fields living on these fixed points.

S^1/Z_2 [T.Kobayashi, *et al.* NPB768(2007)]

Let us consider two states as multiplet.



$$\begin{pmatrix} |x> \\ |y> \end{pmatrix}$$

Then, allowed n-point couplings must be invariant under

- rotational part (twist)

$$i = 1 \sim n$$

$$\begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} = -1 \begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} \ ,$$

- translational part (lattice)

$$\begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} = \sigma_3 \begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} \ ,$$

- permutation between two fixed points

$$\begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} = \sigma_1 \begin{pmatrix} |x^i> \\ |y^i> \end{pmatrix} \ ,$$

these transformations.

S^1/Z_2 [T.Kobayashi, *et al.* NPB768(2007)]

n-point couplings must be invariant under the combinations of such transformations, too.

The product of such generators lead to the eight elements

$$\pm 1, \pm\sigma_1, \pm i\sigma_2, \pm\sigma_3 \ .$$

↓

D_4 symmetry

That is, n-point couplings must be invariant under D_4 symmetry, and the multiplets are doublets of D_4 .

Building blocks

The symmetries of 6-dimensional orbifolds can be obtained by combining symmetries of the following blocks.

[T.Kobayashi, *et al.* $NPB768(2007)$]

orbifold	flavor symmetry	twisted sector	string fundamental states
$\mathbb{S}^1/\mathbb{Z}_2$	$D_4 = S_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	untwisted sector	$\mathbf{1}$
		θ-twisted sector	$\mathbf{2}$
$\mathbb{T}^2/\mathbb{Z}_2$	$(D_4 \times D_4)/\mathbb{Z}_2 = (S_2 \times S_2) \ltimes \mathbb{Z}_2^3$	untwisted sector	$\mathbf{1}$
		θ-twisted sector	$\mathbf{4}$
$\mathbb{T}^2/\mathbb{Z}_3$	$\Delta(54) = S_3 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$	untwisted sector	$\mathbf{1}$
		θ-twisted sector	$\mathbf{3}$
		θ^2-twisted sector	$\bar{\mathbf{3}}$
$\mathbb{T}^2/\mathbb{Z}_4$	$(D_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$	untwisted sector	$\mathbf{1}$
		θ-twisted sector	$\mathbf{2}$
		θ^2-twisted sector	$\mathbf{1}_{A_1} + \mathbf{1}_{B_1} + \mathbf{1}_{B_2} + \mathbf{1}_{A_2}$
$\mathbb{T}^2/\mathbb{Z}_6$	trivial		
$\mathbb{T}^4/\mathbb{Z}_8$	$(D_4 \times \mathbb{Z}_8)/\mathbb{Z}_2$	untwisted sector	$\mathbf{1}$
		θ-twisted sector	$\mathbf{2}$
		θ^2-twisted sector	$\mathbf{1}_{A_1} + \mathbf{1}_{B_1} + \mathbf{1}_{B_2} + \mathbf{1}_{A_2}$
		θ^3-twisted sector	$\mathbf{2}$
		θ^4-twisted sector	$4 \times (\mathbf{1}_{A_1} + \mathbf{1}_{B_1} + \mathbf{1}_{B_2} + \mathbf{1}_{A_2})$
$\mathbb{T}^4/\mathbb{Z}_{12}$	trivial		
$\mathbb{T}^6/\mathbb{Z}_7$	$S_7 \ltimes (\mathbb{Z}_7)^6$	untwisted sector	$\mathbf{1}$
		θ^k-twisted sector	$\mathbf{7}$
		θ^{7-k}-twisted sector	$\bar{\mathbf{7}}$

Table 2: Non-Abelian discrete flavor symmetries of the building blocks.

Z_6-II orbifold models

Let us calculate anomalies of discrete flavor symmetries in heterotic orbifold models. We mainly focus on the $Z_6 - II$ orbifold models.
(Note that, we also considered different orbifold models, so that our results are more generally valid.)

We pay attention to only abelian (transrational) parts of non-abelian discrete flavor symmetries ($D_4 \times Z_n s$).
In the $Z_6 - II$ orbifold, there are three translational symmetries.

$$Z_2^{flavor} \times Z_2^{'flavor} \times Z_3^{flavor}$$

We define anomaly coefficients as

$$\mathcal{A}_{Z_n^{flavor}-G-G} = \frac{1}{n} \sum_{\mathbf{r}^{(f)}} q_n^{(f)} \ell(\mathbf{r}^{(f)}) \ .$$

Z_6-II orbifold models

- KRZ model [Kobayashi, Raby, Zhang, NPB704(2005)]

$$G = SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes SO(10) \otimes SU(2)'$$

G	$\mathbb{Z}_2$	$\mathbb{Z}_2'$	$\mathbb{Z}_3$
SU(4)	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1
$\mathrm{SU(2)_L}$	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1
$\mathrm{SU(2)_R}$	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1
SO(10)	0 mod 2	0 mod 2	$\frac{4}{3}$ mod 2
SU(2)′	0 mod 1	0 mod 1	$\frac{1}{3}$ mod 1

Table 3: Summary of $\mathbb{Z}_n$ anomalies in the KRZ model.

Z_3 is anomalous. But it can be canceled by the GS mechanism.

Anomaly free

Discrete version of the GS mechanism

Consider a discrete transformation. $\left(\alpha = \frac{2\pi}{N}\right)$

$$\Phi \to e^{-i\alpha q_N}\Phi$$

Φ : chiral supermultiplet

$$\text{Anomaly} = -i\mathcal{A}_G[\alpha W^a(G)W_a(G)]_F$$

We can cancel this anomaly by the gauge kinetic term,

$$k_G[SW^a(G)W_a(G)]_F$$

$$S \to S + i\frac{\mathcal{A}_G}{k_G}\alpha$$

$$k_G[SW^a(G)W_a(G)]_F + i\mathcal{A}_G[\alpha W^a(G)W_a(G)]_F$$

Cancel

if we correspondingly shift the dilaton supermultiplet.

(Note that the Kahler potential is invariant.)

For instance, the anomaly cancellation conditions for the SM gauge groups are given by

$$\frac{\mathcal{A}_3}{k_3} = \frac{\mathcal{A}_2}{k_2} = \frac{\mathcal{A}_1}{k_1} = \frac{\mathcal{A}_G}{12}.$$

(It is difficult to build realistic models with higher levels in string theory.)

Z_6-II orbifold models

・BHLR model [Buchmuller, Hamaguchi, Lebedev, Ratz, $NPB785(2006)$]

$$G = SU(3) \otimes SU(2) \otimes SU(4) \otimes SU(2)'$$

G	$\mathbb{Z}_2$	$\mathbb{Z}_2'$	$\mathbb{Z}_3$
SU(3)	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1
SU(2)	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1
SU(4)	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1
SU(2)′	0 mod 1	0 mod 1	$\frac{2}{3}$ mod 1

Table 6: Summary of $\mathbb{Z}_n$ anomalies in the BHLR model.

Z_3 is anomalous. But it can be canceled by the GS mechanism.

→ **Anomaly free**

Empirical relations

By conducting many calculations for various models, we can obtain the following empirical relations.

$$\mathcal{A}_{Z_2^{flavor}-G-G} = \frac{n_2^{anom}}{2} \mod 1$$

$$\mathcal{A}_{Z_3^{flavor}-G-G} = \frac{n_3^{anom}}{3} \mod 1$$

Here, n_2^{anom} and n_3^{anom} are the orbifold parameters and we found these parameters are related to anomalous U(1) .

$$t_{U(1)}^{anom} = k^{anom} V + n_2^{anom} W_2 + n_3^{anom} W_3$$

V : shift
W : wilson lines
λ : lattice vector

Anticipation

- There might be some relation between discrete flavor anomalies and U(1) gauge anomalies.
- Anomaly freedom might be guaranteed by GS mechanism of anomalous U(1).

D_4 flavor symmetry

D_4 has eight elements which can be written as products of the two generators g and h .

$$\mathcal{G}_{D_4} = \{e, g, h, gh, hg, ghg, hgh, ghgh\}.$$

g corresponds to translatiomal part ($Z_2^{flavor}, Z_2^{\prime flavor}$).
h corresponds to parmutational part (non-abelian part).

- If this D_4 flavor symmetry originate from heterotic orbifold compactifications,
- If the empirical relations really exist,

Anomalies corresponding to g are forbidden!!

(Anomalies corresponding to h are future work.)