Anomalies of discrete symmetries

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Who is Takeshi Araki?

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Discrete flavor symmetries

Recently, a lot of models with a flavor symmetry based on a finite group have been proposed (\(S_3, A_4, D_N, Q_N \cdots\)), because it is possible....

\- to reduce the number of free parameters in the Yukawa sector.
\- to realize hierarchical structure of fermion masses.
\- to realize bi-maximal structure of MNS matrix.

\[V_{CKM}\]

small mixing

\text{quark}

\[V_{MNS}\]

bi-large mixing

\text{lepton}
Motivation

There are too many models!

Most of the models are meaningless because a flavor symmetry is only one (if there exist it).

But we have no criterion to distinguish them.

I thought it is important to consider a new criterion.

Anomalies of discrete symmetries.

That is, I want to consider a thing such as gauge anomalies.

\[
\mathcal{A}_{[U(1)_Y]^3} = \sum_f \left[ (Y_L^f)^3 - (Y_R^f)^3 \right] = 0
\]

\[
\mathcal{A}_{[SU(2)_L]^2 - U(1)_Y} = \sum_f Y_L^f \ell_2(r^f) = 0
\]

\[
\mathcal{A}_{[SU(3)_C]^2 - U(1)_Y} = \sum_f Y_L^f \ell_3(r^f) = 0
\]

\[
\mathcal{A}_{[U(1)_Y]^2 - D} = ?
\]

\[
\mathcal{A}_{[SU(2)_L]^2 - D} = ?
\]

\[
\mathcal{A}_{[SU(3)_C]^2 - D} = ?
\]
Motivation

To this end, there are two problems.

1. **How do I define anomalies for discrete symmetries?**

2. **Why are discrete anomalies forbidden?**

I would like to talk about these topics in this presentation.
Let us consider a continuous chiral rotation, \( \psi \rightarrow e^{i\alpha(x)\gamma_5}\psi \), in Euclidean space-time.

\[
Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \mathcal{D}A_\mu \exp \left\{ \int d^4x \ L(\psi, A) \right\}
\]

\[
\mathcal{D}\bar{\psi}\mathcal{D}\psi \rightarrow J^{-1}\mathcal{D}\bar{\psi}\mathcal{D}\psi
\]

\[
J^{-1} = \left\{ \det \int d^4x \varphi_n^\dagger(x) \left[ e^{i\alpha(x)\gamma_5} \right] \varphi_m(x) \right\}^{-2}
\]

\[
e^{i\alpha(x)\gamma_5} \simeq 1 + i\alpha(x)\gamma_5
\]

\[
\simeq \exp \left\{ -2 \sum_n \int d^4x \varphi_n^\dagger(x) [i\alpha(x)\gamma_5]_{ij} \varphi_n(x) \right\}
\]

\[
= \exp \left\{ -i \int d^4x \frac{\alpha(x)}{16\pi^2} \text{Tr} \left[ \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(A) F_{\rho\sigma}(A) \right] \right\}
\]

Hence the current does not conserve at the quantum level.
I want to consider the same calculation for discrete ones. Let us consider a discrete chiral rotation, \( \psi \rightarrow e^{i\alpha\gamma_5} \psi \), in Euclidean space-time. For instance \( \alpha = (2\pi/N)q \).

\[
Z = \int D\bar{\psi} D\psi \; DA_\mu \exp \left\{ \int d^4x \; \mathcal{L}(\psi, A) \right\}
\]

\[
D\bar{\psi} D\psi \rightarrow J^{-1} D\bar{\psi} D\psi
\]

\[
J^{-1} = \left\{ \det \int d^4x \; \varphi_n^\dagger (x) [e^{i\alpha\gamma_5}] \varphi_m (x) \right\}^{-2}
\]

In this case, we can not use an infinitesimal transformation because \( \alpha \) is a discrete parameter.
Discrete anomalies

We can expand the Jacobian as follows.

\[
J^{-1} = \left\{ \det \int d^4 x \, \varphi_n^+(x) \left[ e^{i\alpha \gamma_5} \right] \varphi_m(x) \right\}^{-2} \equiv \{ \det C_{nm} \}^{-2}
\]

\[
C_{nm} = \int d^4 x \, \varphi_n^+(x) \left[ e^{i\alpha \gamma_5} \right] \varphi_m(x) = \delta_{nm} + \int d^4 x \, \varphi_{n,i}^+(x) \left[ i\alpha \gamma_5 \right]_{ij} \varphi_{m,j}(x)
\]

\[
+ \int d^4 x \, \varphi_{n,i}^+(x) \frac{1}{2!} \left[ i\alpha \gamma_5 \right]_{ij} \varphi_{m,j}(x)
\]

\[
+ \int d^4 x \, \varphi_{n,i}^+(x) \frac{1}{3!} \left[ i\alpha \gamma_5 \right]_{ij} \varphi_{m,j}(x) + \cdots
\]

\[
= \delta_{nm} + \tilde{C}_{nm} + \frac{1}{2!} \tilde{C}^2_{nm} + \frac{1}{3!} \tilde{C}^3_{nm} + \cdots
\]

Then we use \( \det A = \exp \{ \text{Tr} \ln A \} \) to obtain

\[
J^{-1} = \det \left\{ \int d^4 x \, \varphi_{n,i}^+(x) \left( e^{i\alpha \gamma_5} \right)_{ij} \varphi_{m,j}(x) \right\}^{-2}
\]

\[
= \exp \left\{ -2 \sum_n \int d^4 x \, \varphi_{n,i}^+(x) \left[ i\alpha \gamma_5 \right]_{ij} \varphi_{n,j}(x) \right\}
\]

\[
= \exp \left\{ -i \int d^4 x \, \frac{\alpha}{16\pi^2} \text{Tr} \left[ e^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} \right] \right\}.
\]
Let us consider non-abelian discrete flavor transformations.

\[ \psi_{L,\alpha} \rightarrow U_{\alpha\beta} \psi_{L,\beta} \equiv (e^{iX})_{\alpha\beta} \psi_{L,\beta} \]
\[ \psi_{R,\alpha} \rightarrow V_{\alpha\beta} \psi_{R,\beta} \equiv (e^{iY})_{\alpha\beta} \psi_{R,\beta} \quad (\alpha, \beta: \text{generations}) \]

These are unitary transformations \((UU^\dagger = VV^\dagger = 1)\).

\[ \det(U) = \det(e^{iX}) = e^{i\eta} \]
\[ \eta, \xi \propto \frac{2\pi}{N} \]
\[ \det(V) = \det(e^{iY}) = e^{i\xi} \]

Let us calculate the Jacobian for these transformations.

\[ J^{-1} = \exp \left\{ \frac{\eta i}{32\pi^2} \int d^4x \ \text{Tr} \left[ \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(L) F_{\rho\sigma}(L) \right] \right\} \]
\[ \times \exp \left\{ -\frac{\xi i}{32\pi^2} \int d^4x \ \text{Tr} \left[ \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(R) F_{\rho\sigma}(R) \right] \right\} \]

Here, \(\eta\) and \(\xi\) correspond to the abelian parts of \(U\) and \(V\).

Therefore, we only have to take into account its abelian parts.
Stringy originated discrete symmetries

We want to consider the situation that discrete flavor anomalies are forbidden. But there is no reason....

We assumed that discrete flavor symmetries originate from string theory.

- In fact, we can derive non-abelian discrete flavor symmetries from heterotic orbifold models. [T.Kobayashi, et al. NPB768(2007)]
- Such discrete symmetries reflect geometrical symmetries of internal space.
- The geometrical operations are embedded into the gauge group.
- There might be some relation between discrete anomalies and that of anomalous U(1) gauge symmetries.

We expected that we might get some constraint for discrete anomalies from anomaly freedom of anomalous U(1) if there exist such a relation.

Divide a 1-dimensional space by a 1-dimensional lattice.

Identify the points which are related by a reflection.

The points of both ends are invariant under $Z_2$-twist.

There are brane matter fields living on these fixed points.
Let us consider two states as multiplet.

\[
\begin{pmatrix}
|x> \\
|y>
\end{pmatrix}
\rightarrow
\begin{pmatrix}
|x>
\\
|y>
\end{pmatrix}
\]

Then, allowed \(n\)-point couplings must be invariant under:

- rotational part (twist)
  \[
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  -1 & 0 \\
  0 & -1
  \end{pmatrix}
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix}
  =
  -1
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix},
  \quad i = 1 \sim n
  \]

- translational part (lattice)
  \[
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  1 & 0 \\
  0 & -1
  \end{pmatrix}
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix}
  =
  \sigma_3
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix},
  \]

- permutation between two fixed points
  \[
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix}
  \rightarrow
  \begin{pmatrix}
  0 & 1 \\
  1 & 0
  \end{pmatrix}
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix}
  =
  \sigma_1
  \begin{pmatrix}
  x^i > \\
  y^i >
  \end{pmatrix},
  \]

these transformations.
n-point couplings must be invariant under the combinations of such transformations, too. The product of such generators lead to the eight elements 

$$\pm 1, \pm \sigma_1, \pm i\sigma_2, \pm \sigma_3$$.

That is, n-point couplings must be invariant under $D_4$ symmetry, and the multiplets are doublets of $D_4$. 

**Building blocks**

The symmetries of 6-dimensional orbifolds can be obtained by combining symmetries of the following blocks.


<table>
<thead>
<tr>
<th>orbifold</th>
<th>flavor symmetry</th>
<th>twisted sector</th>
<th>string fundamental states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^1/Z_2$</td>
<td>$D_4 = S_2 \times (Z_2 \times Z_2)$</td>
<td>untwisted sector, $\theta$-twisted sector</td>
<td>1, 2</td>
</tr>
<tr>
<td>$T^2/Z_2$</td>
<td>$(D_4 \times D_4)/Z_2 = (S_2 \times S_2) \times Z_2^2$</td>
<td>untwisted sector, $\theta$-twisted sector</td>
<td>1, 4</td>
</tr>
<tr>
<td>$T^2/Z_3$</td>
<td>$\Delta(54) = S_3 \times (Z_3 \times Z_3)$</td>
<td>untwisted sector, $\theta$-twisted sector, $\theta^2$-twisted sector</td>
<td>1, 3, 3</td>
</tr>
<tr>
<td>$T^2/Z_4$</td>
<td>$(D_4 \times Z_4)/Z_2$</td>
<td>untwisted sector, $\theta$-twisted sector, $\theta^2$-twisted sector</td>
<td>1, 2, $1_{A_1} + 1_{B_1} + 1_{B_2} + 1_{A_2}$</td>
</tr>
<tr>
<td>$T^2/Z_6$</td>
<td>trivial</td>
<td>untwisted sector, $\theta$-twisted sector, $\theta^2$-twisted sector</td>
<td>1, 2</td>
</tr>
<tr>
<td>$T^4/Z_8$</td>
<td>$(D_4 \times Z_8)/Z_2$</td>
<td>untwisted sector, $\theta$-twisted sector, $\theta^2$-twisted sector, $\theta^3$-twisted sector, $\theta^4$-twisted sector</td>
<td>$1_{A_1} + 1_{B_1} + 1_{B_2} + 1_{A_2}$, 2, $4 \times (1_{A_1} + 1_{B_1} + 1_{B_2} + 1_{A_2})$</td>
</tr>
<tr>
<td>$T^4/Z_{12}$</td>
<td>trivial</td>
<td>untwisted sector, $\theta$-twisted sector, $\theta^2$-twisted sector, $\theta^3$-twisted sector, $\theta^4$-twisted sector</td>
<td>1, 7</td>
</tr>
<tr>
<td>$T^5/Z_7$</td>
<td>$S_7 \times (Z_7)^6$</td>
<td>untwisted sector, $\theta$-twisted sector, $\theta^7$-twisted sector</td>
<td>1, 7</td>
</tr>
</tbody>
</table>

Table 2: Non-Abelian discrete flavor symmetries of the building blocks.
**Z$_6$-II orbifold models**

Let us calculate anomalies of discrete flavor symmetries in heterotic orbifold models. We mainly focus on the $Z_6 - II$ orbifold models. (Note that, we also considered different orbifold models, so that our results are more generally valid.)

We pay attention to only abelian (transrational) parts of non-abelian discrete flavor symmetries ($D_4 \times Z_n$).

In the $Z_6 - II$ orbifold, there are three translational symmetries.

$$Z_2^{\text{flavor}} \times Z_2^{\prime \text{flavor}} \times Z_3^{\text{flavor}}$$

We define anomaly coefficients as

$$A_{Z_n^{\text{flavor}} - G - G} = \frac{1}{n} \sum_{r(f)} q_n^{(f)} \ell(r^{(f)}) .$$
**Z₆-II orbifold models**

- KRZ model  [Kobayashi, Raby, Zhang, *NPB* 704 (2005)]

\[ G = SU(4) \otimes SU(2)_L \otimes SU(2)_R \otimes SO(10) \otimes SU(2)' \]

<table>
<thead>
<tr>
<th>G</th>
<th>( \mathbb{Z}_2 )</th>
<th>( \mathbb{Z}_2' )</th>
<th>( \mathbb{Z}_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(4)</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>( \frac{1}{3} ) mod 1</td>
</tr>
<tr>
<td>SU(2)_L</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>( \frac{1}{3} ) mod 1</td>
</tr>
<tr>
<td>SU(2)_R</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>( \frac{1}{3} ) mod 1</td>
</tr>
<tr>
<td>SO(10)</td>
<td>0 mod 2</td>
<td>0 mod 2</td>
<td>( \frac{4}{3} ) mod 2</td>
</tr>
<tr>
<td>SU(2)'</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>( \frac{1}{3} ) mod 1</td>
</tr>
</tbody>
</table>

Table 3: Summary of \( \mathbb{Z}_n \) anomalies in the KRZ model.

\( Z_3 \) is anomalous. But it can be canceled by the GS mechanism.

**Anomaly free**
Consider a discrete transformation.

Gauged groups are given by

For instance, the anomaly cancellation conditions for the SM

if we correspondingly shift the dilaton supermultiplet.

\[
\begin{align*}
\frac{1}{1} & = \frac{1}{1} = \frac{1}{2} = \frac{1}{2} \\
\end{align*}
\]

Note that the Kähler potential is invariant.

( It is difficult to build realistic models with higher levels in string theory. )

\[
\begin{align*}
\mathcal{F} \left[ (\mathcal{G}) \mathcal{W} \mathcal{M} \mathcal{W} \mathcal{W} \right] & + \mathcal{F} \left[ (\mathcal{G}) \mathcal{W} \mathcal{M} \mathcal{S} \right] \\
\mathcal{F} & \left[ (\mathcal{G}) \mathcal{G} \mathcal{W} \mathcal{W} \left[ (\mathcal{G}) \mathcal{W} \mathcal{W} \mathcal{W} \right] \right] \\
\mathcal{F} & \left[ (\mathcal{G}) \mathcal{W} \mathcal{W} \mathcal{W} \right] \\
\mathcal{F} & \left[ (\mathcal{G}) \mathcal{W} \mathcal{W} \mathcal{W} \right] \\
\mathcal{F} & \left[ (\mathcal{G}) \mathcal{W} \mathcal{W} \mathcal{W} \right]
\end{align*}
\]

We can cancel this anomaly by the charge kinetic term,

\[
\begin{align*}
\mathcal{F} \left[ (\mathcal{G}) \mathcal{W} \mathcal{M} \mathcal{W} \mathcal{W} \mathcal{W} \right] & + S \\
& \rightarrow S
\end{align*}
\]

Consider a discrete transformation.

Discrete version of the GS mechanism
**$Z_6$-II orbifold models**

- BHLR model [Buchmüller, Hamaguchi, Lebedev, Ratz, *NPB*785(2006)]

$$G = SU(3) \otimes SU(2) \otimes SU(4) \otimes SU(2)'$$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\mathbb{Z}_2$</th>
<th>$\mathbb{Z}_2'$</th>
<th>$\mathbb{Z}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(3)$</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$SU(2)$</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$SU(4)$</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$SU(2)'$</td>
<td>0 mod 1</td>
<td>0 mod 1</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

Table 6: Summary of $\mathbb{Z}_n$ anomalies in the BHLR model.

$Z_3$ is anomalous. But it can be canceled by the GS mechanism.

**Anomaly free**
Empirical relations

By conducting many calculations for various models, we can obtain the following empirical relations.

\[ \mathcal{A}_{Z_2^{flavor}} - G - G = \frac{n_2^{anom}}{2} \mod 1 \]
\[ \mathcal{A}_{Z_3^{flavor}} - G - G = \frac{n_3^{anom}}{3} \mod 1 \]

Here, \( n_2^{anom} \) and \( n_3^{anom} \) are the orbifold parameters and we found these parameters are related to anomalous U(1).

\[ t^{anom}_{U(1)} = k^{anom} V + n_2^{anom} W_2 + n_3^{anom} W_3 \]

Anticipation

- There might be some relation between discrete flavor anomalies and U(1) gauge anomalies.
- Anomaly freedom might be guaranteed by GS mechanism of anomalous U(1).
**D₄ flavor symmetry**

D₄ has eight elements which can be written as products of the two generators g and h.

\[ G_{D₄} = \{ e, g, h, gh, hg, ghg, hgh, ghgh \}. \]

g corresponds to translational part ( \( Z^\text{flavor}_2, Z'_\text{flavor} \)).
h corresponds to parmutational part ( non-abelian part ).

- If this \( D₄ \) flavor symmetry originate from heterotic orbifold compactifications,
- If the empirical relations really exist,

Anomalies corresponding to g are forbidden!!

( Anomalies corresponding to h are future work. )