# Lattice Operator Product Expansion and the Structure Functions


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**Based on**

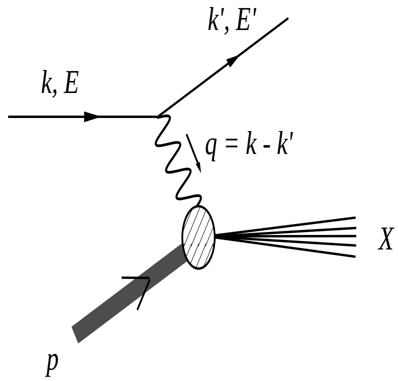
W. Detmold and C-JDL, Phys Rev D73, 014501 (2006).

# Outline

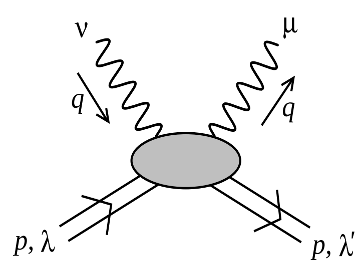
- Hadronic tensor and the OPE.
- The OPE on the lattice.
- Extracting moments from lattice data.
- Application to the pion distribution amplitude.
- Conclusion.

# Deeply Inelastic Lepton-Hadron Scattering, I

A good review: Aneesh Manohar, hep-ph/9204208.




Dimensionless hadronic tensor $W_S^{\mu\nu}(p,q)$
$\longrightarrow$ decomposed into the structure functions $F_{1,2}$ and $g_{1,2}$.




Dimensionless hadronic tensor $T_S^{\mu\nu}(p,q)$
$\longrightarrow$ decomposed into the structure functions $\tilde{F}_{1,2}$ and $\tilde{g}_{1,2}$.

- Optical theorem relates Imag of $\tilde{F}, \tilde{g}$ to $F$, $g$.
- The structure functions are functions of $x = -q^2/2p \cdot q$ and $-q^2/p^2$.

# Deeply Inelastic Lepton-Hadron Scattering, II

- DIS is the study of the regime $-q^2/p^2 \to \infty$ at fixed $x = -q^2/2p\cdot q$.

  $\longrightarrow$ Asymptotic freedom of QCD.

  $\longrightarrow$ The Bjorken scaling.

- Physical region is $0 \leq x \leq 1$.

- The DIS regime can be shown to be dominated by the structure of the hadron along the light-cone.

  $\longrightarrow$ Difficult for field theory formulated in Euclidean space.

- Can perform an operator product expansion.

  $\longrightarrow$ A short distance expansion, works at $x \to \infty$.

  $\longrightarrow$ Extract information, *i.e.*, moments of the structure functions, in this unphysical region.

# Hadronic tensor and the OPE

$$W_S^{\mu\nu}(p,q) = \int d^4z \, \mathrm{e}^{iq\cdot z} \langle p,S| \left[ J^\mu(z), J^\nu(0) \right] |p,S\rangle$$

↓ Imaginary part of

$$T_S^{\mu\nu}(p,q) = \int d^4z \, \mathrm{e}^{iq\cdot z} \langle p,S| T \left[ J^\mu(z) J^\nu(0) \right] |p,S\rangle$$

↓ the OPE

$$T\left[ J^\mu(z) J^\nu(0) \right] = \sum \mathcal{C}_i(z^2,\mu^2) z_{\mu_1} \ldots z_{\mu_n} \, \mathcal{O}_i^{\mu\nu\mu_1\ldots\mu_n}(\mu)$$

twist = dimension - spin

Powers of $1/x$ in the matrix elements.

# Lattice calculations

- Analytic continuation

  $\rightarrow$ Difficult to obtain $T_S^{\mu\nu}$ directly.

- Operator mixing and renormalisation

  $\rightarrow$ Difficult for high-spin operators.

# The OPE on the lattice
## *General features*

$$\underbrace{\langle p,S|T\left[J^\mu(z)J^\nu(0)\right]|p,S\rangle}_{\text{Simulation}} = \underbrace{\sum \mathcal{C}_i(z^2,\mu^2)\; z_{\mu_1}\ldots z_{\mu_n}}_{\text{Analytical calculation}}\; \underbrace{\langle p,S|\mathcal{O}_i^{\mu\nu\mu_1\ldots\mu_n}(\mu)|p,S\rangle}_{\text{Fits}}$$

- First investigated in kaon physics.

C. Dawson *et al.*, 1998.

# Our proposal

- Simulation of $\sum_S T_S^{\mu\nu}$ $\longrightarrow$ Continuum limit.

  $\longrightarrow$ No power divergence.

- Perform the OPE in Euclidean space
- Fit the matrix elements.

  $\longrightarrow$ No need for analytic continuation.

  $\longrightarrow$ No need for operator matching.

- Not obtaining $T_S^{\mu\nu}$ in Minkowski space directly.

# The OPE on the lattice
## *Specific features*

- A fictitious "valence" heavy quark $\Psi$ and current

$$J^{\mu}_{\Psi,\psi}(z) = \bar{\Psi}(z)\gamma^{\mu}\psi(z) + \bar{\psi}(z)\gamma^{\mu}\Psi(z).$$

- Study the Euclidean Compton scattering tensor

$$T^{\mu\nu}_{\Psi,\psi} \;=\; \sum_{S}\int d^4z \;\; \mathrm{e}^{iq\cdot z}\left\langle p,S\left|T\left[J^{\mu}_{\Psi,\psi}(z)J^{\nu}_{\Psi,\psi}(0)\right]\right|p,S\right\rangle$$

- Sum the target-mass effects.

- Compute the twist-two matrix elements.

## Why a "valence" heavy quark?

- Two large scales, $q^2$ and $m_{\Psi}$.

$$\Lambda_{\mathrm{QCD}} \ll m_{\Psi} \sim \sqrt{q^2} \ll \frac{1}{a}$$

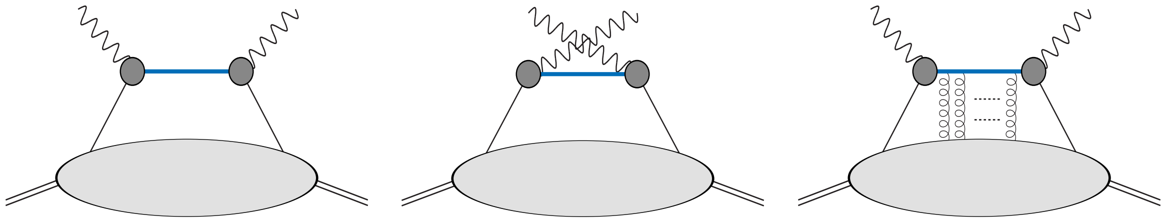
- Remove many higher-twist contributions.

- No all-to-all propagator in the simulation.

- The Fourier transform is practical $\rightarrow z_4 \sim 1/m_{\Psi}$.

# The OPE on the lattice

*Some details*



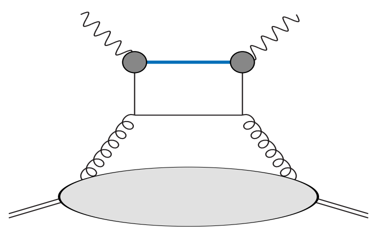
Higher-twist components

$$\frac{i(i\not{D}+\not{q})+m_\Psi}{(iD+q)^2+m_\Psi^2} = -\frac{i(i\not{D}+\not{q})+m_\Psi}{Q^2+D^2-m_\Psi^2}\sum_{n=0}^{\infty}\left(\frac{-2i\ q\cdot D}{Q^2+D^2-m_\Psi^2}\right)^n$$

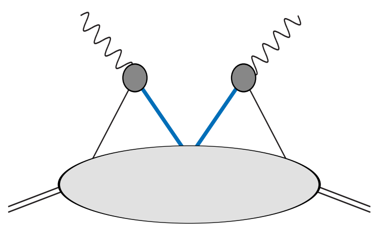
$$\frac{\Lambda^2_{\mathrm{QCD}}}{q^2+m_\Psi^2}$$

$$Q^2 = -q^2, \qquad M_\Psi = m_\psi + \alpha/2$$

$$Q^2 + D^2 - m_\Psi^2 = \tilde{Q}^2 = Q^2 - M_\Psi^2 + \alpha M_\Psi + \beta$$



No contribution in $T^{\mu\nu}_{\Psi,u} - T^{\mu\nu}_{\Psi,d}$.



Removed because $\Psi$ is non-dynamical.

# Extrating moments from data

## *The Euclidean Compton tensor*

$$\sum_S \left\langle p,S \left| \bar{\psi}\gamma^{\{\mu_1}(iD^{\mu_2})\ldots\left(iD^{\mu_n\}}\right) - \mathrm{tr} \right| p,S\right\rangle = A_\psi^n(\mu^2)\left[p^{\mu_1}\ldots p^{\mu_n} - \mathrm{tr}\right]$$

$$\sum_S \left\langle p,S \left| \bar{\psi}\left(iD^{\{\mu_1}\right)\ldots\left(iD^{\mu_n\}}\right) - \mathrm{tr} \right| p,S\right\rangle = \widehat{A}_\psi^n(\mu^2)\left[p^{\mu_1}\ldots p^{\mu_n} - \mathrm{tr}\right]$$

$$T_{\Psi,\psi}^{\{\mu\nu\}}(p,q) = i\sum_{\substack{n=2\\ \mathrm{even}}}^{\infty} A_\psi^n(\mu^2)\ \zeta^n\ \mathcal{F}\left[C_n^{(1)}(\eta), C_{n-1}^{(2)}(\eta), C_{n-2}^{(3)}(\eta), n, q^2, \tilde{Q}^2, \mu^2\right]$$

$$-2i\frac{M(m_\Psi - m)}{\tilde{Q}^2}\delta^{\mu\nu}\sum_{\substack{n=0\\ \mathrm{even}}}^{\infty}\widehat{\mathcal{C}}_n\ \widehat{A}_\psi^n(\mu^2)\ \zeta^n\ C_n^{(1)}(\eta)$$

- The Gegenbauer polynomial: target-mass effects.

$$\eta = \frac{p\cdot q}{\sqrt{p^2 q^2}}$$

- $\tilde{Q}^2 \sim -q^2 - M_\Psi^2$ is the large scale for the OPE.

$$\zeta = \frac{\sqrt{p^2 q^2}}{\tilde{Q}^2}$$

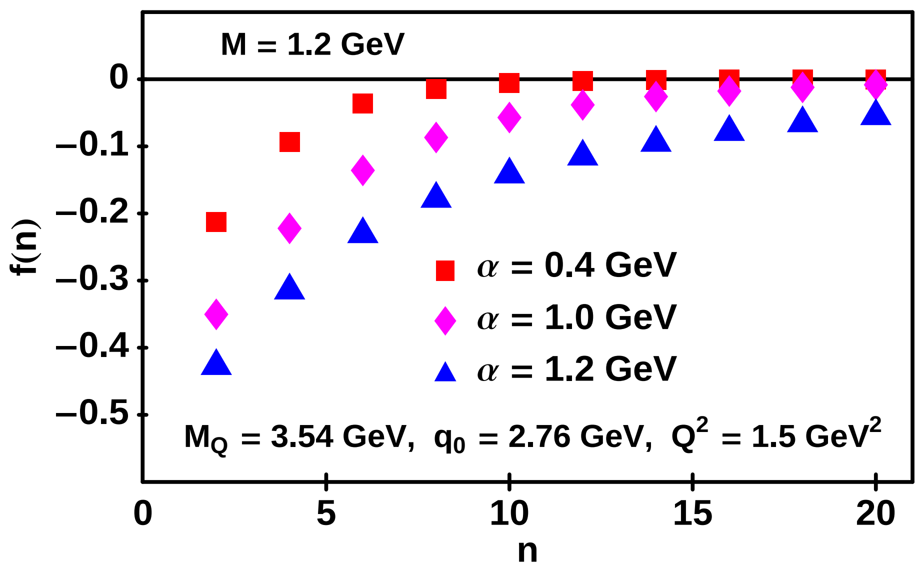
- Remove $\widehat{A}_\psi^n$ by choosing $\mu \neq \nu$.

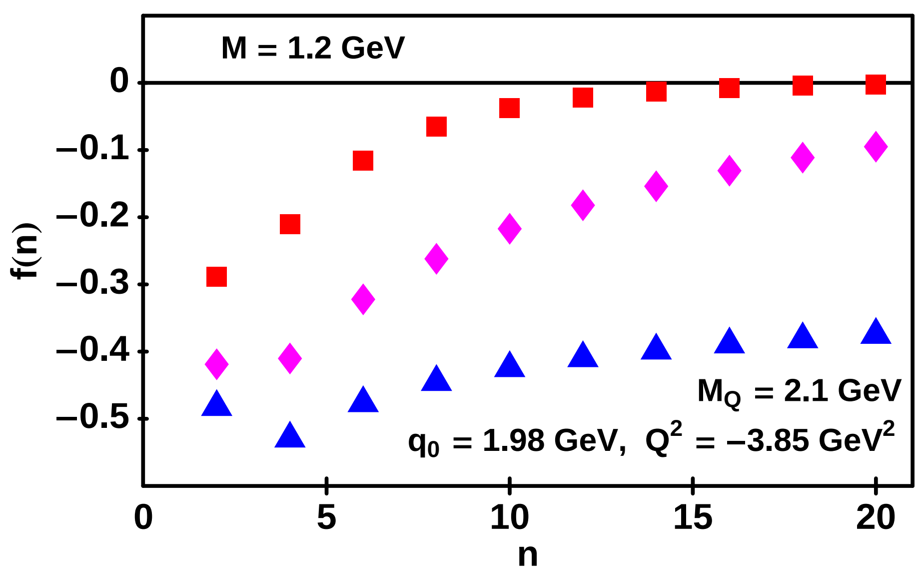
# Extracting moments from data

## *Plots*

$$T^{\{34\}}_{\Psi,\psi}(p,q) = \sum_{n=2,\text{even}}^{\infty} A^n_\psi(\mu^2) f(n)$$

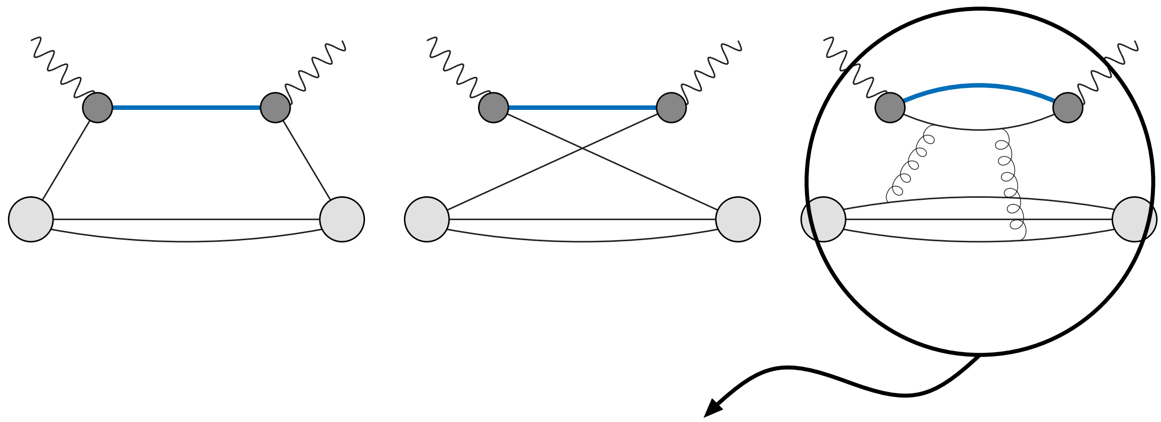
$$p = (0,0,0,iM) \quad , \quad q = (0,0,\sqrt{q_0^2 - Q^2}, iq_0)$$



# Extracting moments from data

*Correlators*



No contribution for $T^{\mu\nu}_{\Psi,u} - T^{\mu\nu}_{\Psi,d}$

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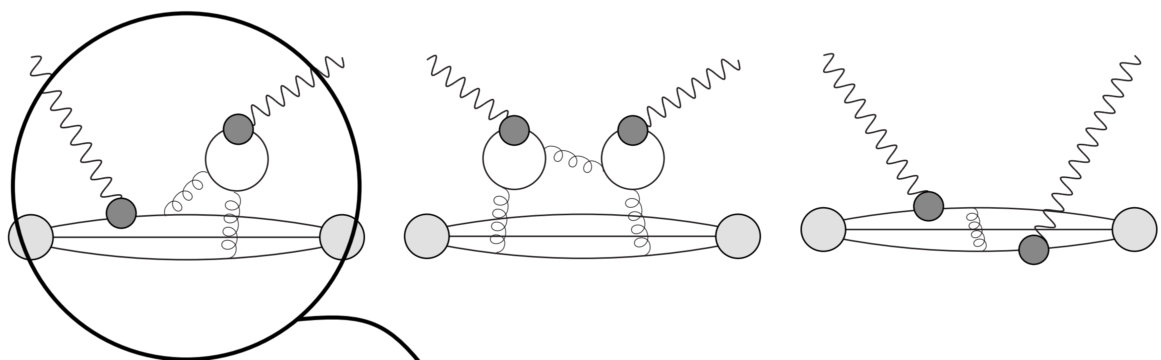
Additional contractions if light-light current is used.



Contributes to $T^{\mu\nu}_{\Psi,u} - T^{\mu\nu}_{\Psi,d}$

# Application to the Pion Distribution Amplitude

Related to the matrix element $\langle \pi | T \left[ \bar{d}(z) \gamma_\mu \gamma_5 u(0) \right] | 0 \rangle$.

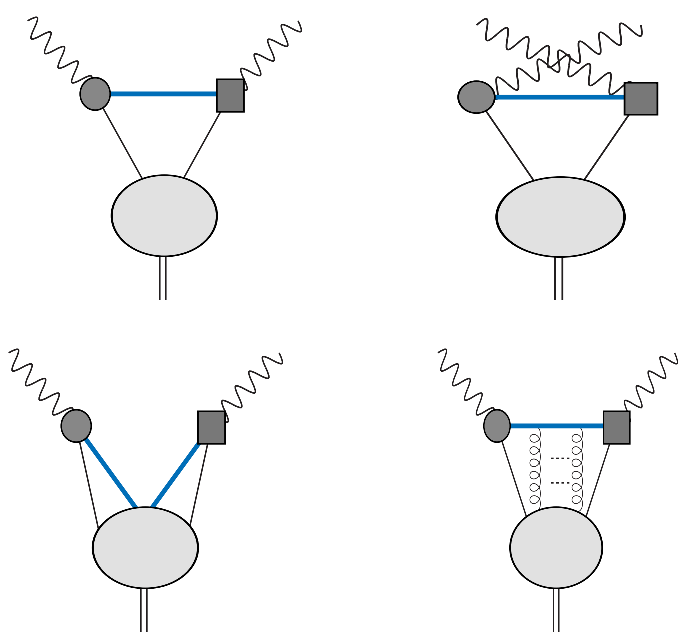
- A crucial input in $B \to \pi\pi$ decays via QCD factorisation.

- The OPE leads to the need of the matrix elements

$$\left\langle \pi(p) \left| \bar{\psi} \gamma^{\{\mu_1} \gamma_5 \left( iD^{\mu_2} \right) \ldots \left( iD^{\mu_n\}} \right) \right| 0 \right\rangle.$$

- These can be obtained by applying the OPE to the "unphysical" matrix element

$$\left\langle \pi(p) \left| T \left[ V^\mu_{\Psi,\psi}(z) A^\nu_{\Psi,\psi}(0) \right] \right| 0 \right\rangle.$$



# Conclusion

- Extract moments via the OPE on the lattice.
- Can be applied to other nucleon structure functions.
- Can be applied to the pion distribution amplitude
- Numerical work is being carried out.