Nov. 6, 2007.

① 分 Midterm 考卷, 解释考題, 提醒: ~1.5 hr

② Group theory 簡介 - I ~1.5 hr.

A good reference:

Lie Algebras in particle physics, by Howard Georgi

* Semi-Simple Lie Algebras and their representations

by Robert N. Cahn (already put on line)

* definition of group. H.W. 4: 1, 2, 5

* order, subgroup, homomorphism, isomorphism

representation,

equivalent, irreducible.

group generator for continuous group.

* definition of $SU(n)$, $U(n)$, $SO(n)$, $O(n)$ . . .

930314 王繼文

930302 李政憲

931704 魏瑀嫺

921027 張凱維

930344 王棋斌

930336 陳柏榮

930343 蕭[illegible]綱

930353 李映潔

921118 陳[illegible]

930360 潘維[illegible]

930308 林慧婷

930313 姚欣佳

Set 1

A set $\mathcal{G}$ is called a group if the following 4 axioms held: ①

a map $\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ composition law

$$\begin{cases} T_i\cdot T_j \in \mathcal{G}, \text{ if } T_i \,\&\, T_j \in \mathcal{G} \\ E\cdot T = T\cdot E = T \quad \forall T\in\mathcal{G} \\ T\cdot T^{-1} = T^{-1}\cdot T = E, \quad \forall T\in\mathcal{G} \\ (T_i\cdot T_j)\cdot T_k = T_i\cdot(T_j\cdot T_k) \end{cases}$$

* The # of elements in $\mathcal{G}$ is called the order of $\mathcal{G}$

S_3 : order $=6$. for translation : order $=\infty$

* If a subset S of $\mathcal{G}$ is a group, then S is a subgroup of $\mathcal{G}$

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* ~~$((T_i\cdot T_j)\cdot T_k\cdot T_l$~~

transformation : in mind :

A specific set of representation

a simple example

2 objects : Z_2

	S	E ← second
S	E	S
E	S	E

↑ act first

multiplication table

$S=-1,\ E=1$, • real # multiply

$S=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, • matrix multiplication

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$\hat{\Gamma}(S)=-1,\ \hat{\Gamma}(E)=1$

$\Gamma(S)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Gamma(E)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

representation S in representation Γ

infinite # of representation

Abelian:

$T_i\cdot T_j = T_j\cdot T_i$

$\forall\, T_i, T_j \in \mathcal{G}$

represent by a $d \times d$ matrix

$\{ \Gamma(T_1), \Gamma(T_2) \cdots \}$

$$\Gamma(T_1) \cdot \Gamma(T_2) = \Gamma(T_1 \cdot T_2)$$

matrix multiplication

A: ~~in vertable matrix~~ invertible matrix

$\tilde{\Gamma}(T_i) = A^{-1} \Gamma(T_i) A$

another representation.

$$\tilde{\Gamma}(T_1) \cdot \tilde{\Gamma}(T_2) = A^{-1}\Gamma(T_1) A A^{-1} \Gamma(T_2) A = A^{-1}\Gamma(T_1 \cdot T_2) A$$
$$= \tilde{\Gamma}(T_1 \cdot T_2)$$

Any two representations related by a ~~sim~~ change of basis are called equivalent.

ϕ: a mapping of $\mathcal{G}$ (group) onto $\mathcal{G}'$ (group)

$$\phi(T_1) \cdot \phi(T_2) = \phi(T_1 \cdot T_2), \quad \forall\, T_1, T_2 \in \mathcal{G}$$

we call ϕ a group homomorphism (homomorphism)

$\mathcal{G}$ & $\mathcal{G}'$ are homomorphic

e.g. $\Gamma_1(S) = 1$, $\Gamma_1(E) = 1$ → not faithful rep

~~isomorphism~~

If the map is 1–1, we say $\mathcal{G}'$ & $\mathcal{G}$ are isomorphic & ϕ an isomorphism

We call the set of elements $T \in \mathcal{G}$ such that

$\phi(T) = E$, the kernel of ϕ.

e.g. $\Gamma_2(S) = -1$

$\Gamma_2(E) = 1$

→ rep is faithful

if choose $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\tilde{\Gamma}(S) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{\Gamma}(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \tilde{\Gamma}(T_i) = A^{-1}\Gamma(T_i)A$

$\Rightarrow \tilde{\Gamma}(S) = \begin{pmatrix} \Gamma_1(S) & 0 \\ 0 & \Gamma_2(S) \end{pmatrix}$, $\tilde{\Gamma}(E) = \begin{pmatrix} \Gamma_1(E) & 0 \\ 0 & \Gamma_2(E) \end{pmatrix}$

In general: one can always form the following ~~form~~ from many different representation.

$$\Gamma(T_i) = \begin{pmatrix} \tilde{\Gamma}_1(T_i) & & 0 \\ & \tilde{\Gamma}_2(T_i) & \\ 0 & & \ddots \end{pmatrix}$$

$\Gamma_1(S) = 1$
$\Gamma_1(E) = 1$
~~not~~ isomorphic

$\Gamma_2(S) = -1$
$\Gamma_2(E) = 1$
isomorphic

$\Rightarrow$ a reducible representation

$\Updownarrow$

irreducible representation

If a rep is not reducible ~~to~~ any further to a block form is

~~rep~~ $\Rightarrow$ rep is provided by isomorphism
$\Rightarrow$ the rep is faithful

Irreducible, ~~equivalent~~ Inequivalent represent.

$\Rightarrow$ A finite group. $\Rightarrow$ a finite #

of irreducible inequivalent rp.

- Privacy & Security

CI-9.1

given M_1 & N_1. How do we check ~~the~~ if

$$M_1 = A^{-1} N_1 A \quad ?$$

$$N_1 |i\rangle = \lambda_i |i\rangle$$

$$|\hat{i}\rangle = A^{-1}|i\rangle$$

$$M_1 |\hat{i}\rangle = A^{-1} N_1 A A^{-1} |i\rangle = \lambda_i A^{-1}|i\rangle = \lambda_i |\hat{i}\rangle$$

$\Rightarrow$ eigenvalue ~~invariant~~ not changed

$$N_1^n |i\rangle = \lambda_i^n |i\rangle$$

$$\mathrm{Tr}(N_1) = \sum_{i=1}^{d} \lambda_i$$

$$\mathrm{Tr}(N_1^n) = \sum_{i=1}^{d} \lambda_i^n \quad \Rightarrow \quad \mathrm{tr}(N_1^n) = \mathrm{tr}(M_1^n) \quad \forall n$$

a set of matrices, another set of matrices $\{M_1, M_2\}$, $\{N_1, N_2\}$ is $\underline{M_1 = A^{-1} N_1 A, \text{ and } M_2 = A^{-1} N_2 A}$?

need to check not only

$$\left.\begin{array}{l} \mathrm{tr}(M_1^n) = \mathrm{tr}(N_1^n) \\ \mathrm{tr}(M_2^n) = \mathrm{tr}(N_2^n) \end{array}\right\} \text{not enough}$$

$\because$ M_2 could be $\hat{A}^{-1} N_2 \hat{A}$, $\hat{A} \neq A$

also $\mathrm{tr}((M_1 M_2)^n) = \mathrm{tr}((N_1 N_2)^n)$

actually $\mathrm{tr}(M_1 \cdot M_2 \cdots M_1 \quad) = \mathrm{tr}(N_1 \cdot N_2 \cdots N_1 \cdots)$

any sequence

5

vacuum bubbles created in one vacuum phase evolve and die in a different vacuum background. This suggests a new possibility of quantum vacuum polarization via the creation and annihilation of whole domains of spacetime in which the energy density is different from that of the ambient spacetime. As a matter of fact, the novelty of our field model is the onset of a new type of ``Higgs mechanism for membranes" triggered solely by quantum fluctuations. The effect of such fluctuations can be accounted for by an effective potential. As in Umezawa's approach, this effective potential is consistent with the dynamical generation of a bag with surface tension out of the vacuum.

Next Up Previous
Next: Bibliography **Up:** Membrane Vacuum as a **Previous:** 4. Dynamics of the

Stefano Ansoldi
Department of Theoretical Physics
University of Trieste
TRIESTE - ITALY

If 2 representations are equivalent

$\Rightarrow \mathrm{Tr}(\Gamma(T_i)) = \mathrm{tr}(\tilde{\Gamma}(T_i))$

we say Γ & $\tilde{\Gamma}$ are equivalent.

$\mathrm{Tr}(\Gamma(T_i))$ is the character of group element T_i in rep Γ.

Charact

Set-2

How many reps are there?

Infinite # reducible

$$M \,| r^i \rangle = \lambda_i | r^i \rangle$$

$tr(M) = \sum_{i=1}^{d} \lambda_i$ to characterization a group

Not useful.

Infinite # of group element of continuous group

(translation / rotation)

we can change the transformation smoothly

$f(x) \to f(x+\epsilon) = (1+\epsilon \frac{d}{dx}) f(x)$

can build up a finite transform from infinite

N translation, each translate by $\frac{a}{N}$, $N \to \infty$, a: fixed

$$\left(1+\frac{a}{N}\frac{d}{dx}\right)^N f(x)$$

$U = \left(1+\frac{a}{N}\frac{d}{dx}\right)^N$, $\quad \ln U = N \log\left(1+\frac{a}{N}\frac{d}{dx}\right)$

$$= N\left[\frac{a}{N}\frac{d}{dx} + O\left(\frac{1}{N^2}\right)\right]$$

$$= a\frac{d}{dx} + O\left(\frac{1}{N}\right)$$

$$\Rightarrow U = \exp\left(a\frac{d}{dx}\right)$$

$$\left(1+\frac{a}{N}\frac{d}{dx}\right)^N f(x) = e^{a\frac{d}{dx}} f(x) = \left(1 + a\frac{d}{dx} + \frac{a^2}{2!}\frac{d^2}{dx^2} + \cdots\right) f(x)$$

$$= f(x+a)$$

$T(a) = e^{a\frac{d}{dx}} = e^{iaP}$, $\quad P = -i\frac{d}{dx}$

$e^{i\alpha\hat{O}}$, → normalization still be one.

→ unitary

$$(e^{i\alpha O})^{\dagger} = (e^{i\alpha O})^{-1}$$

$$e^{-i\alpha O^{\dagger}} = e^{-i\alpha O} \Rightarrow O^{\dagger} = O$$

hermitian operator

p: the generator of translation.

group element → group generator
infinite # (finite #)

need $\underline{\underline{i}}$ to make it hermitian!

A rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ +s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

infinitesimal rotation, $\theta \to \epsilon$, $c_\epsilon = 1 - \epsilon^2 \ldots$
$s_\epsilon = \epsilon + O(\epsilon^3)$

$$\rightarrow \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\epsilon \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$: generator of the rotation along $\hat{z}$-axis

$$R(\theta) = e^{i\theta T} \qquad T^2 = \mathbb{1}$$

$$= \mathbb{1}\cos\theta + i\sin\theta T = \begin{pmatrix} c_\theta & 0 \\ 0 & c_\theta \end{pmatrix} + i s_\theta \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}$$

$$\boxed{\exp(i\alpha\,\text{generator}) = \text{group element}}$$

$$T(\vec{a}) = \exp(i a_x \hat{p}_x + i a_y \hat{p}_y + i a_z \hat{p}_z)$$

3 generators

3 parameters

$$\boxed{N \text{ parameters} \leftrightarrow N \text{ generators}}$$

$\rightarrow$ irreducible, inequivalent, generator

$A(e^{i\alpha T})A^{-1}$ similarity transformation on the group element $\longrightarrow$ group generators

$$= A\left(1 + i\alpha T - \frac{\alpha^2}{2!}T^2 + \dots\right)A^{-1}$$

$$= AA^{-1} + i\alpha ATA^{-1} - \frac{\alpha^2}{2!}ATA^{-1}ATA^{-1} \dots)$$

$$= 1 + i\alpha T' - \frac{\alpha^2}{2!}(T'^2) \dots$$

$$= e^{i\alpha T'} \qquad \boxed{T' = ATA^{-1}}$$

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reducible

$$T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \\ & & & \ddots \end{pmatrix} \qquad T^n = \begin{pmatrix} T_1^n & 0 & 0 \\ 0 & T_2^n & 0 \\ 0 & 0 & T_3^n \end{pmatrix}$$

group element reducible = generator reducible

" $e^{i\alpha T}$ [illegible] T^n matter

· multiplication table of $\longrightarrow$ translation parameter

$$T(x_1', x_2' \ldots x_n')\, T(x_1, x_2 \ldots x_n)$$

$$= T\left(f_1(x_1' x_2', \ldots x_n', x_1, x_2 \ldots),\ f_2(x_1, x_2 \ldots), \ldots\ f_n(x_1 \ldots) \right)$$

* $f(x_1, x_2 \ldots x_{n-1}', x_n')$ are smooth function of x_n' & x_n'
  $\Rightarrow$ focusing on the Lie group
* $T(x_1 = 0\ 0\ 0 \ldots 0) = \mathbb{1}$

$$f_j(x_1 \ldots x_n, \underbrace{0, 0 \ldots 0}_{n}) = x_j \quad (1)$$

$$f_j(\underbrace{0 \ldots 0}_{n}, x_1' \ldots x_n') = x_j' \quad (2)$$

$$f_i = c_i + d_{ij} x_j + e_{ij} x_j' + \underline{x^2 + x'^2 \ldots}$$

$x_j' = 0$, $\quad d_{ij} = \delta_{ij}$, $\quad c_i = 0$

$x_j = 0$, $\quad c_i = 0$, $\quad e_{ij} = \delta_{ij}$

$$\Rightarrow f_i = x_i + x_i' + c_{ijk} x_j x_k' + d_{ijk} \cancel{x_j x_k} + e_{ijk} \cancel{x_j' x_k'}$$

$x_j = 0$, $\quad x_j' = 0$

$x = 0 \rightarrow$ this can't contribute, so $e_{ijk} = 0$

$x' = 0$, this can't contribute, so $d_{ijk} = 0$

$x_j' = 0 \rightarrow x'$

this must be zero

$x_j' = 0$

$\Rightarrow$ $\underline{d_{ijk} = 0}$

$$\Rightarrow f_i = x_i + x_i' + c_{ijk} x_j x_k' + O(x^2 x', x'^2 x)$$

rep expand around the identity

$$\Gamma\left(T(x_1 \cdots x_n)\right) = \mathbb{1} + \boxed{i}\, x_a T_a + \frac{1}{2} x_a x_b T_{ab} = \left(\frac{T_a T_b + T_b T_a}{2}\right)$$

need i again this is symmetric symmetric must be

$$\Gamma\left(T(x_1' \cdots x_n')\right) = \mathbb{1} + i x_a' T_a + \frac{1}{2} x_a' x_b' T_{ab}$$

$$\left(\mathbb{1} + i x_a T_a + \frac{1}{2} x_a x_b T_{ab} \cdots\right)\left(\mathbb{1} + i x_a' T_a + \frac{1}{2} x_a' x_b' T_{ab}\right)$$

$$= \mathbb{1} + i(x_a T_a + x_a' T_a) + \frac{1}{2}(x_a x_b + x_a' x_b') T_{ab}$$

$$- x_a x_b' T_a T_b$$

$$= \Gamma\left(T(f_1(x_1 \cdots, x_1' \cdots x_n'), f_2(x, x') \cdots)\right)$$

$$= \mathbb{1} + i f_a T_a + \frac{1}{2} f_a f_b T_{ab}$$

$$= 1 + i(x_a + x_a' + c_{abc} x_b x_c') T_a + \frac{1}{2}(x_a + x_a' + \cdots c_{a\theta\phi} x_\theta x_\phi')$$
$$\times (x_b + x_b' + \cdots c_{b\theta\phi} x_\theta x_\phi') T_{ab}$$

$$= \mathbb{1} + i x_a T_a + i x_a' T_a + \frac{1}{2} x_a x_b T_{ab} + \frac{1}{2} x_a' x_b' T_{ab} + i c_{abc} x_b x_c' T_a$$
$$+ x_a x_b' T_{ab}$$

another important identity is Jacobi identity

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0$$

$$\Rightarrow \quad -T_a T_b = T_{ba} + i T_c c_{cab}$$

$$(T_a T_b - T_b T_a)^\dagger$$
$$= -T_a^\dagger T_b^\dagger + T_b^\dagger T_a^\dagger = -[T_a, T_b]$$

therefore i to make f_{abc} real

$$\Rightarrow \quad \boxed{[T_a, T_b] = i f_{abc} T_c}$$

$$f_{abc} = (c_{cab} - c_{cba})$$

multiplication table $\Rightarrow$ generator Lie algebra

$\Rightarrow$ group structure const

remark

expand to higher order ;

$\Rightarrow$ once order 2 is satisfied

higher order is OK !

$O(2)$ isomorphic group of 2×2 real orthogonal matrices

$$A^T A = A A^T = \mathbb{1}$$

closure ? $A_1 A_1^T = \mathbb{1}$, $A_2 A_2^T = \mathbb{1}$

does $A_1 A_2$ belong to the class

$$(A_1 A_2)^T A_1 A_2 = A_2^T A_1^T A_1 A_2 = \mathbb{1}$$

$\det(A) = \det(A^T)$, $\det(AB) = \det(A)\det(B)$

$(\det(A))^2 = \mathbb{1}$ $\quad \det A = \pm 1$

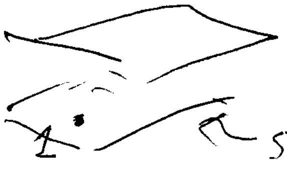
$+1$: normal rotation $\quad$ proper rotation

-1 : parity on $\quad$ improper rotation

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A, \quad AA^T = 1$

$\det(A) = -1$ $\quad$ any A with $\det(A) = -1$

can be written as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times$ proper rotation



$\leftarrow SO(2)$ $\det = +1$

$O(2)$ is disconnected

contains 2 sub classes

vector

$$\sum_{i=1}^{n} u_i v_i = \langle u | v \rangle \qquad v_i \to \sum_{j=1}^{n} A_{ij} v_j$$

$$|v\rangle \to A|v\rangle$$

$$\langle u|v\rangle \to \langle u|A^T A|v\rangle \qquad \langle v| \to \langle v|A^T$$

$$\Rightarrow \quad A^T A = \mathbb{1} \quad \rightarrow O(N)$$

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dirdim

$\epsilon_{i_1 i_2 \cdots i_d}$. total anti symmetric

$$\epsilon_{12\cdots d} = +1$$

$$\epsilon_{i_1, i_2 \cdots i_d} \; u_{i_1} v_{i_2} \cdots w_{i_d}$$

$$= \epsilon_{i_1 i_2 \cdots i_d} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_d j_d} \; v_{j_1} u_{j_2} \cdots w_{j_d}$$

$$\epsilon_{i_1 i_2 \cdots i_d} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_d j_d} \qquad v \; u \cdots w$$

$$= \epsilon_{i_2 i_1 \cdots i_d} A_{i_2 j_1} A_{i_1 j_2} \cdots A_{i_d j_d} \qquad v \; u \quad w$$

$$= -\epsilon_{i_1 i_2 \cdots} A_{i_1 j_1} A_{i_2 j_2} \cdots \qquad v \; u \quad w.$$

is totally asymmetric ; just need one #1 to fit it

$$\Rightarrow \epsilon_{i_1 i_2 \cdots i_d} A_{i_1 1} A_{i_2 2} \cdots A_{i_d d} = \det(A)$$

extra condition.

$\epsilon_{i_1 i_2 i_3} u_{i_1} v_{i_2} w_{i_3}$ also invariant

$$= \epsilon_{i_1 i_2 i_3} A_{i_1 j_1} A_{i_2 j_2} A_{i_3 j_3} v_{j_1} v_{j_2} w_{j_3} = \epsilon_{j_1 j_2 j_3} v_{j_1} v_{j_2} w_{j_3}$$

$\epsilon_{i_1 i_2 i_3} A_{i_1 j_1} A_{i_2 j_2} A_{i_3 j_3}$

$= \epsilon_{i_1 i_2 i_3} A_{i_2}$

$\epsilon_{i \cdots} A \quad = \det(A)$

$\Rightarrow \det(A) = 1$

$\Rightarrow SO(d)$ group

$O(d) \quad A^T A = 1 \qquad U(d)$

$\downarrow$

$d \times d$ matrix

$SO(d) \qquad SU(d)$

$\downarrow$ special $\det(A) = 1$

$A = e^{i\alpha T}$

$A^T = e^{i\alpha T^T} = A^{-1} = e^{-i\alpha T}$

$\Rightarrow \boxed{T^T = -T}$ form

antisymmetry & hermitian

$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \rightarrow$ pure imaginary.

$\begin{array}{c|c} & d \\ \hline i & \begin{bmatrix} & i & \\ -i & & \end{bmatrix} d \end{array} \rightarrow O(d).$ # of generators $\frac{d \times (d-1)}{2}$