Int. to HEP. (I)
Nov. 13, 2007

* Angular momentum in QM
* spin & SU(2)
* C-G coefficients (Clebsch-Gordan)

HW: chap 4:
11, 12 ← angular moment adding
19, 20, 21, 23 ← SU(2) project

* orbital Angular momentum in QM.

$$\hat{L} = \hat{r} \times \hat{p} \quad , \quad \hat{p} = -i\hbar \vec{\partial}$$

e.g. $L_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$

properties:

① L^2 & L_z ~~to be~~ commute with hamiltonian.

② eigenvalues: $l(l+1)\hbar^2$, $m_l\hbar$

$$\underbrace{m_l = -l, -l+1, \cdots -1, 0, +1, \cdots l-1, l}_{\text{total } (2l+1) \text{ possibilities}}$$

S^2, S_z → $s(s+1)\hbar^2$, $m_s\hbar$

$$\underbrace{m_s = -s, -s+1, \cdots s-1, s}_{(2s+1) \text{ possibilities}}$$

* Spin: An intrinsic angular momentum degrees of freedom.

The classical picture of spining sphere is helpful but can't go very far.

e.g. consider the spin of an electron $(s=\frac{\hbar}{2})$ by a dimensional analysis:

$m_e r_e v \sim \hbar$ ⇐ angular momentum

$E = m_e c^2 \sim \frac{e^2}{r_e}$ ⇐ mass ~ EM field

$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$

$\Rightarrow r_e \sim \frac{e^2}{m_e c^2} \Rightarrow \frac{e^2 v}{c^2} \sim \hbar$

$v \sim \frac{c^2\hbar}{e^2} \sim 137c \gg c$

⇒ the origin of spin ??

when you get an electron by quantizing an electron field, the field already carries spin.

~~total angular momentum $\vec{J} = L + S$~~

Bosons	spin-0 :	Higgs	, π, K - - (pseudo scalar mesons.)	
	spin-1 :	force mediators	, vector mesons ρ, K^*, ϕ, ω - -	
	spin-2 :	graviton	, ——	$\longrightarrow$ can't find a renormalizable theory
Fermions	spin-$\frac{1}{2}$:	quarks, leptons,	Baryon octet .	
	spin-$\frac{3}{2}$:	gravitino	, Baryon decuplet	classically, can't find a field theory for $s > 2$.

※ Addition of angular momenta.

labeled by a ket : $|l\, m\rangle$ or $|s\, m_s\rangle$

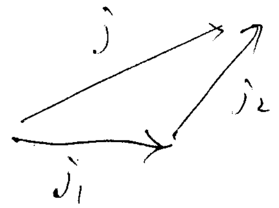
total angular momentum, $\vec{J} = L + S$

How to add? classically, just add up the $\vec{L}$ vectors.

In QM,

$$|j_1, m_1\rangle + |j_2\, m_2\rangle$$

$\Rightarrow$ m still add, $\Rightarrow$ $m = m_1 + m_2$



$j_1 \parallel j_2 \Rightarrow j = j_1 + j_2$

$j_1 \uparrow\downarrow j_2 \Rightarrow j = |j_1 - j_2|$

but again j still need to be quantized :

$$j = |j_1 - j_2|,\ |j_1 - j_2| + 1,\ \cdots\cdots,\ (j_1 + j_2)$$

More than 2 $\vec{J}$? Add them up one by one -

* C-G coefficients

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{j=|j_1-j_2|}^{(j_1+j_2)} C^{j\ j_1\ j_2}_{m\ m_1\ m_2} |j\,m\rangle, \quad \text{with} \quad m = m_1 + m_2$$

e.g. ~~the~~ an e^- in a hydrogen atom occupies the orbital state $|2,-1\rangle$ and the spin state $|\frac{1}{2},\frac{1}{2}\rangle$.

$$\ell + s = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2} \qquad m = -1 + \frac{1}{2} = -\frac{1}{2}$$

$$|1,-1\rangle |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{5}}\, |\tfrac{5}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{3}{5}}\, |\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$\uparrow \qquad\qquad \Uparrow$$

40% $J = \frac{5}{2}$ 60% $J = \frac{3}{2}$

read horizontally ⇨

read vertically ⇩

$$|j\,m\rangle = |\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{3}{5}}\, |2,0\rangle |\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{5}}\, |2,-1\rangle |\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

e.g. 2 spin-$\frac{1}{2}$ state, $\quad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2}\tfrac{1}{2}\rangle = |1\,1\rangle$$

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}}\, |1,0\rangle + \sqrt{\tfrac{1}{2}}\, |0\,0\rangle$$

$$|-\tfrac{1}{2}\rangle |\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{2}}\, |1,0\rangle - \sqrt{\tfrac{1}{2}}\, |0,0\rangle$$

$$|-\tfrac{1}{2}\rangle |-\tfrac{1}{2}\rangle = |1,-1\rangle$$

or

$$|1\,1\rangle = |\tfrac{1}{2}\rangle |\tfrac{1}{2}\rangle$$

$$|1,0\rangle = \sqrt{\tfrac{1}{2}}\, |\uparrow\downarrow\rangle + \sqrt{\tfrac{1}{2}}\, |\downarrow\uparrow\rangle$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle$$

$$|0,0\rangle = \sqrt{\tfrac{1}{2}}\, |\uparrow\downarrow\rangle - \tfrac{1}{\sqrt{2}}\, |\downarrow\uparrow\rangle$$

spin - $\frac{1}{2}$

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$ $\Rightarrow$ called spinor

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} \qquad |\alpha|^2 : \uparrow \qquad |\beta|^2 : \downarrow$$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

eigenvalues of $\hat{S}_x = \pm\frac{\hbar}{2}$, eigenstates : $\chi_\pm = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \pm\frac{1}{\sqrt{2}}\end{pmatrix}$

$\Rightarrow$ $\begin{pmatrix}\alpha\\\beta\end{pmatrix} = a\chi_+ + b\chi_-$, $a = \frac{1}{\sqrt{2}}(\alpha+\beta)$ $\quad b = \frac{1}{\sqrt{2}}(\alpha-\beta)$

$\hat{S}_x^2 = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\Rightarrow$ eigenvalues is always $\frac{\hbar^2}{4}$.

Same for $\hat{S}_y^2$ and $\hat{S}_z^2$ $\Rightarrow$ $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3}{4}\hbar^2 = \frac{1}{2}(\frac{1}{2}+1)\hbar^2$

$\Rightarrow$ Pauli matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

and $\hat{S} = \frac{\hbar}{2}\vec{\sigma}$

spinor under rotation

$$U^{+}(\phi)\ \vec{S}\ U(\phi) = R_{ij}\ \vec{S}$$

$$R_{ij} = \delta_{ij} + \epsilon_{mij}\,\delta\phi_m$$

$$U(\phi) = \exp\left(-\frac{i}{\hbar}\phi\cdot\hat{a}\right)$$

recall $(x, y) \to (x - y\,\delta\phi_z,\ y + x\,\delta\phi_z)$

$$U(\delta\phi) \simeq \mathbb{1} - \frac{i}{\hbar}\delta\phi\cdot\hat{a}$$

or $R_{ji} = \sigma_{ji} + \epsilon_{mji}\,\delta\phi_m$

$$(\delta_{ji} + \epsilon_{mji}\delta\phi_m)\left(\mathbb{1} + \frac{i}{\hbar}\delta\phi\cdot\hat{a}\right)\hat{\sigma}_i\left(\mathbb{1} - \frac{i}{\hbar}\delta\phi\cdot\hat{a}\right) \simeq \hat{\sigma}_j$$

$$\Rightarrow (\delta_{ji} + \epsilon_{mji}\,\delta\phi_m)\left(\hat{\sigma}_i + \frac{i}{\hbar}\delta\phi_\alpha[\hat{a}_\alpha, \hat{\sigma}_i]\right)$$

$$\Rightarrow \hat{\sigma}_j + \epsilon_{mji}\,\delta\phi_m\hat{\sigma}_i + \frac{i}{\hbar}\delta\phi_\alpha[\hat{a}_\alpha, \hat{\sigma}_j] \simeq \hat{\sigma}_j$$

or $\quad \frac{i}{\hbar}[\hat{a}_\alpha, \hat{\sigma}_j] = -\epsilon_{\alpha j i}\hat{\sigma}_i \quad ; \quad \frac{1}{\hbar}[\hat{a}_\alpha, \hat{\sigma}_j] = i\,\epsilon_{\alpha j i}\,\hat{\sigma}_i$

the only solution $\Rightarrow \hat{a}_\alpha = \frac{\hbar}{2}\hat{\sigma}_\alpha$ $\qquad$ recall $\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \epsilon_{ijk}\frac{\sigma_k}{2}$

$$\Rightarrow \underset{\text{Rotation}}{U(\phi)} = \underset{\text{orbital angular}}{U_L(\phi)}\,U_{\text{spin}}(\phi) = \exp\left[-\frac{i}{\hbar}\vec{\phi}\cdot\left(\hat{L} + \underbrace{\frac{\hbar}{2}\vec{\sigma}}_{\vec{S}}\right)\right]$$

$$\hat{\vec{J}} = \hat{L} + \hat{S}$$

or spinor alone

$$U(\phi) = \exp\left(-\frac{i}{2}\vec{\phi}\cdot\vec{\sigma}\right)$$

rotate 4π to return the original one.

$$L_x = yp_z - zp_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

- $[L_i, L_j] = i\,\epsilon_{ijk} L_k$

- $L_\pm = L_x \pm i L_y$, $[L_+, L_-] = 2L_z$; $[L_z, L_\pm] = \pm L_\pm$

- $L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2$

$$= L_-L_+ + L_z(L_z+1) = L_+L_- + L_z(L_z-1)$$

- $[L^2, L_z] = [L^2, L_+] = [L^2, L_-] = 0$

- $L_zL_+ - L_+L_z = L_+$ or $L_zL_+ = L_+(L_z+1)$

$L_z|j,m\rangle = m|j,m\rangle$, $L_zL_+|j,m\rangle = (m+1)\underbrace{L_+|j,m\rangle}_{\sim |j,m+1\rangle}$

- L^2 acts on $|j,j\rangle$

$\Rightarrow L^2|j,j\rangle = L_-L_+ + L_z(L_z+1)|j,j\rangle = (0 + j(j+1))|j,j\rangle$

$\Rightarrow L^2$ has an eigenvalue $j(j+1)$

- $L_+L_-|j,m\rangle = [L^2 - L_z(L_z-1)]|j,m\rangle = [j(j+1) - m(m-1)]|j,m\rangle$

$L_-L_+|j,m\rangle = [L^2 - L_z(L_z+1)]|j,m\rangle = [j(j+1) - m(m+1)]|j,m\rangle$

$$|L_+|j,m\rangle|^2 = \langle j,m|L_-L_+|j,m\rangle = [j(j+1) - m(m+1)]\langle j,m|j,m\rangle \geq 0$$

$$|L_-|j,m\rangle|^2 = [j(j+1) - m(m-1)]\langle j,m|j,m\rangle \geq 0$$

$\Rightarrow$ $-j \leq m \leq j$, and $L_+|j,j\rangle = L_-|j,-j\rangle = 0$

$\Rightarrow$ $L_+|j,m\rangle = \sqrt{j(j+1) - m(m+1)}\;|j,m+1\rangle$

$L_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\;|j,m-1\rangle$

An example of calculating the CG coefficients for $1 \times \frac{1}{2}$

$$1 \otimes \frac{1}{2} = \frac{3}{2} + \frac{1}{2}$$

$$L_- = L_{1-} + L_{2-}$$

$$\boxed{|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = |1,1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle}$$

$L_- \downarrow$

$$\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{1(1+1)-1(1-1)}\,|1,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|1,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{2}\,|1,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |1,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

or $$\boxed{|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

$L_- \downarrow$

$$\underbrace{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}}_{2}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\left(\sqrt{1(1+1)-0(0-1)}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + 1\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)$$
$$+\sqrt{\tfrac{1}{3}}\left(\sqrt{2}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + 0\right)$$

or $$2\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\left(\sqrt{2}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right) + \sqrt{\tfrac{2}{3}}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\Rightarrow \boxed{|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

$L_- \downarrow$

$$\underbrace{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-(-\tfrac{1}{2})(-\tfrac{1}{2}-1)}}_{\sqrt{3}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \sqrt{\tfrac{1}{3}}\left(0 + 1\,|1\,-1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)$$
$$+\sqrt{\tfrac{2}{3}}\left(\sqrt{1(1+1)-0(0-1)}\,|1\,-1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + 0\right) = \sqrt{3}\,|1\,-1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\Rightarrow |\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = |1\,-1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$|\frac{1}{2},\frac{1}{2}\rangle = a\,|1\,1\rangle|\frac{1}{2}\,-\frac{1}{2}\rangle + b\,|1\,0\rangle|\frac{1}{2}\,\frac{1}{2}\rangle$ $\Rightarrow$ must be orthogonal to $|\frac{3}{2}\,\frac{1}{2}\rangle$ and $|a|^2+|b|^2=1$

$$\Rightarrow |\tfrac{1}{2}\,\tfrac{1}{2}\rangle = -\sqrt{\tfrac{2}{3}}\,|1\,1\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$L_- \downarrow$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\sqrt{\tfrac{2}{3}}\left(\sqrt{2}\,|1,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right) + \sqrt{\tfrac{1}{3}}\left(\sqrt{2}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right)$$
$$= \sqrt{\tfrac{2}{3}}\,|1\,-1\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|1\,0\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

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