



SUSY QM for S.H.O.

& Hydrogen atom.

S.H.O.

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} \omega^2 x^2$$

$$a \equiv \frac{1}{\sqrt{2\omega}} \frac{d}{dx} + \sqrt{\frac{\omega}{2}} x, \quad a^\dagger \equiv -\frac{1}{\sqrt{2\omega}} \frac{d}{dx} + \sqrt{\frac{\omega}{2}} x$$

$$\Rightarrow H = (a^\dagger a + \frac{1}{2}) \omega$$

also

$$[a, a^\dagger] = 1, \quad [H, a] = -a\omega, \quad [H, a^\dagger] = a^\dagger\omega$$

$\therefore$  a adjoint to  $a^\dagger \Rightarrow$  eigenvalue of  $a^\dagger a$  can't be negative.

$$\langle a^\dagger a \rangle \geq 0 \quad \text{or} \quad |a|\psi_0\rangle|^2 = 0$$

for lowest eigenvalue

$$\Rightarrow \left( \frac{1}{\sqrt{2\omega}} \frac{d}{dx} + \sqrt{\frac{\omega}{2}} x \right) \psi_0 = 0$$

$$\text{or } \psi_0 = \left( \frac{\omega}{2\pi} \right)^{\frac{1}{4}} e^{-\frac{\omega}{2} x^2}, \quad \omega a^\dagger a |\psi_0\rangle = (H - \frac{\omega}{2}) |\psi_0\rangle = 0$$

$$\text{or } H |\psi_0\rangle = \frac{\omega}{2} |\psi_0\rangle = E_0 |\psi_0\rangle$$

~~$$H a^\dagger |\psi_0\rangle = \omega (a^\dagger a + \frac{1}{2}) a^\dagger |\psi_0\rangle$$~~

number operator

$$\hat{n} \equiv a^\dagger a$$

$$[\hat{n}, a] = \hat{n}a - a\hat{n} = -a$$

$$[\hat{n}, a^\dagger] = \hat{n}a^\dagger - a^\dagger\hat{n} = a^\dagger$$

~~$$a^\dagger$$~~

~~$$a^\dagger a^\dagger |\psi_0\rangle = a^\dagger a^\dagger |\psi_0\rangle = 0$$~~

$$\hat{n} a^\dagger |\psi_0\rangle = a^\dagger (\hat{n} + 1) |\psi_0\rangle = a^\dagger |\psi_0\rangle \quad \text{or } \psi_1$$

$$\hat{n}^2 a^\dagger |\psi_1\rangle = \hat{n} a^\dagger a^\dagger |\psi_0\rangle = a^\dagger (\hat{n} + 1) a^\dagger |\psi_0\rangle = a^\dagger (2a^\dagger + a^\dagger \hat{n}) |\psi_0\rangle = 2 a^\dagger a^\dagger |\psi_0\rangle \quad \text{or } \psi_2$$

$$\hat{n}^m (a^\dagger)^m |\psi_0\rangle = m (a^\dagger)^m |\psi_0\rangle \quad \text{or } \psi_m$$

normalized to  $|\psi_m\rangle = \frac{(a^\dagger)^m}{\sqrt{m!}} |\psi_0\rangle$

$$E_m = (m + \frac{1}{2}) \omega, \quad m = 0, 1, 2, \dots$$

$a a^\dagger = 1 + a^\dagger a$

$$H a^\dagger |\psi_0\rangle = \omega (a^\dagger a + \frac{1}{2}) a^\dagger |\psi_0\rangle = \omega (\hat{n} + \frac{1}{2}) a^\dagger |\psi_0\rangle = \frac{3}{2} \omega a^\dagger |\psi_0\rangle$$

$$|a^\dagger |\psi_0\rangle|^2 = \langle \psi_0 | a a^\dagger |\psi_0\rangle = \langle \psi_0 | (1 + a^\dagger a) |\psi_0\rangle = \langle \psi_0 | \psi_0\rangle = 1$$

$$|a^{\dagger 2} |\psi_0\rangle|^2 = \langle \psi_0 | a a^\dagger a^\dagger |\psi_0\rangle = \langle \psi_0 | a (1 + a^\dagger a) a^\dagger |\psi_0\rangle = \langle \psi_0 | a a^\dagger a^\dagger |\psi_0\rangle = 2 \langle \psi_0 | \psi_0\rangle = 2$$

for the Coulomb potential.

(2)

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} R + V(r)R = E R$$

$$R = \frac{u}{r} \quad \frac{1}{r^2} \left( r^2 \left( \frac{u}{r} \right)' \right)' = \frac{1}{r^2} \left( r^2 \left( \frac{u'}{r} - \frac{u}{r^2} \right) \right)'$$

$$= \frac{1}{r^2} \left( r u' - u \right)' = \frac{1}{r^2} \left( u' + r u'' - u' \right) = \frac{u''}{r}$$

$$\Rightarrow \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - \frac{e^2}{r} \right] u = E u$$

$$\text{or} \left[ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2me^2}{\hbar^2 r} \right] u = \frac{2mE}{\hbar^2} u$$

can we cast it into the following form?

$$\Rightarrow \left( -\frac{d}{dr} - w \right) \left( \frac{d}{dr} - w \right) u = \frac{2mE}{\hbar^2} u$$

$$w = a + \frac{b}{r}$$

$$-\frac{d^2}{dr^2} + w^2 + w'$$

$$E' = \frac{2mE}{\hbar^2} + a^2$$

$$w^2 + w' = a^2 + \frac{2ab}{r} + \frac{b^2}{r^2} - \frac{b}{r^2} = \frac{b(b-1)}{r^2} + \frac{2ab}{r} + a^2$$

extra constant.  
that's fine.

$$\Rightarrow b = l+1, \quad a = -\frac{me^2}{\hbar^2(l+1)}$$

$$\Rightarrow \text{ground state energy} = 0, \quad \text{or} \quad 0 = \frac{2mE}{\hbar^2} + a^2 = \frac{2mE}{\hbar^2} + \frac{m^2 e^4}{\hbar^4 (l+1)^2}$$

$$E' = 0 = \frac{2m}{\hbar^2} \left( E + \frac{me^4}{2\hbar^2(l+1)^2} \right)$$

$$\text{or} \quad E = -\frac{me^4}{2\hbar^2(l+1)^2}$$

$$\left( \frac{d}{dr} - w \right) u = 0 \quad \Rightarrow \quad \left( \frac{d}{dr} + \frac{me^2}{\hbar^2(l+1)} - \frac{l+1}{r} \right) u = 0$$

$$\frac{d}{dr} \ln u = \frac{l+1}{r} - \frac{me^2}{\hbar^2(l+1)}$$

$$\text{or} \quad d \ln u = (l+1) \frac{dr}{r} - \frac{me^2}{\hbar^2(l+1)} dr$$

$$u_0 \propto r^{l+1} \exp\left(-\frac{me^2 r}{\hbar^2(l+1)}\right), \quad \text{or} \quad R \propto r^l \exp\left(-\frac{me^2}{\hbar^2(l+1)} r\right)$$

An interesting feature of writing physics in this way

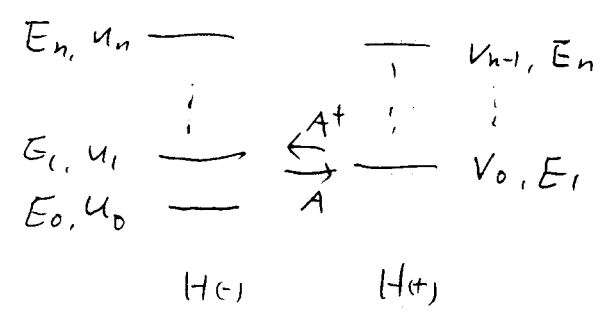
$$H_{(-)} = A^+ A, \quad A^+ = -\frac{d}{dr} - W, \quad A = \frac{d}{dr} - W$$

$$[A, A^+] = -2\frac{d}{dr} W = -2W'$$

$$H_{(+)} = AA^+ = A^+ A - 2W' = H_{(-)} - 2W'$$

Say if we have found out all the solutions  $u_i$  &  $E_i$  for  $H_{(-)}$ ,

Then  $H_{(-)} u_i = E_i u_i$



$$H_{(+)} A u_n = \underline{AA^+} A u_n = A H_{(-)} u_n = E_n \underline{A} u_n$$

$\Rightarrow A u_n$  is an eigenfunction of  $H_{(+)}$  with eigenvalue  $E_n$

~~Similarly~~,  $\Rightarrow V_{n+1} = \frac{1}{\sqrt{E_n}} A u_n$

~~H(-)~~ Similarly  $A^+$  bring  $V_{n-1}$  to  $u_n$  and they are degenerated.

So, let's ~~try~~ look at  $H_{(+)}$  for Coulomb problem

$$H_{(+)} = AA^+ = A^+ A - 2W' = \left( -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2me^2}{\hbar^2 r} \right) - 2 \left( -\frac{l+1}{r^2} \right)$$

$$= \left( -\frac{d^2}{dr^2} + \frac{(l+1)(l+2)}{r^2} - \frac{2me^2}{\hbar^2 r} \right)$$

The ~~diff~~ transformation can be read,  $b = l+1 \rightarrow b = l+2,$

So,  $E_{(+)}^{(0)} = -\frac{me^4}{2\hbar^2 (l+2)^2}$ , and  $R_{(+)}^{(0)} \propto r^{l+1} \exp\left(-\frac{me^2}{\hbar^2 (l+2)} r\right)$

or  ~~$R_{(-)}^{(1)}$~~   $R_{(-)}^{(1)} \propto A^+ \left( r^{l+1} \exp\left(\frac{-me^2}{\hbar^2 (l+2)} r\right) \right) = \left( -\frac{d}{dr} + \frac{l+2}{r} - \frac{me^2}{\hbar^2 (l+2)} \right) \left( r^{l+1} \exp\left(\frac{-me^2}{\hbar^2 (l+2)} r\right) \right)$   
 $\propto r^l \exp\left(\frac{-me^2}{\hbar^2 (l+2)} r\right)$

⇒ 主量子数  $n = n_r + l + 1$

$$\boxed{E_n = -\frac{me^4}{2\hbar^2 n^2}} = -\frac{13.6 \text{ eV}}{n^2} \quad n = 1, 2, 3, \dots$$

~~l~~ = 0, 1, 2, ...  
s p d f ...

for a given  $n = n_r + l + 1$   $l = (n-1), \dots, 0$ ,

the degeneracy is therefore

$$\sum_{l=0}^{n-1} (2l+1) = \sum_{l=0}^{n-1} 1 + 2 \sum_{l=0}^{n-1} l = \cancel{n} + 2 \frac{(n-1)n}{2} = n^2$$

$$E_{\text{photon}} = h\nu = E_{n_i} - E_{n_f} = -\frac{me^2}{2\hbar^2} \left( \frac{1}{n_i^2} - \frac{1}{n_f^2} \right)$$

930308 林慧婷

930313 姚欣佑

930349 李日泰

930343 蕭子國

930353 李映潔

912118 陳子乞

930360 潘欣華

921027 張凱維

931704 魏瑞嫻

