Scattering Matrices and Magnon Bound states in Gauge/String Correspondence

Heng-Yu Chen

DAMTP, University of Cambridge

Based on hep-th/0605155, 0608047 and 0610265 with Nick Dorey (DAMTP) and Keisuke Okamura (University of Tokyo)

Outline of the Seminar

1. Basics of AdS/CFT.
2. Recap on spin chains and spinning strings.
3. Scattering matrices in gauge and string theories.
4. The magnon bound states in gauge theory.
5. The giant and dyonic giant magnons.
6. The scattering of magnon bound states.
7. Conclusion and work in progress.

Basics of AdS/CFT

- Exact equivalence between $\mathcal{N} = 4$ Super Yang-Mills in the planar limit and free IIB string theory in $AdS_5 \times S^5$. (Maldacena 1997)
- Arise from the two different descriptions of low energy/decoupling limit of coinciding D3- branes.
- Basic Tests: Identifications of Global Symmetry Groups:
  - $SO(2,4)$: Isometry Group of AdS_5 = Conformal Group of Four-Dimensional QFT.
  - $SO(6) \cong SU(4)$: Isometry Group of S^5 = R-Symmetry Group of $\mathcal{N} = 4$ SYM.
  - $PSU(2,2|4)$: Symmetry Enhancement in the near horizon geometry of $AdS_5 \times S^5$, $16 \longrightarrow 32$ fermions of IIB = 32 superconformal symmetries of $\mathcal{N} = 4$ SYM.
- It is a realization of Holographic Principle: *"In ultimate theory of Quantum Gravity, physics within some volume (AdS_5) should be encoded by some theory at its boundary ($\mathcal{N} = 4$ SYM), so that its entropy satisfies the Bekenstein bound."* ('t Hooft).

- Different Regimes of the correspondence is parameterized by t' Hooft coupling

$$\lambda = g_{YM}^2 N = g_s N \,, \quad N \to \infty \;\; (\mathrm{Planar-Limit})$$

- The planar gauge theory is perturbative when when $\lambda \ll 1$, loop expansion reliable.
- The string sigma model is perturbative when $\lambda \gg 1$, or $R \gg l_s$ gravity is reliable.
- Useful Strong-Weak Type Duality: But Difficult to Prove or Disprove!!
- By AdS/CFT: *The spectrum of scaling dimensions for gauge invariant operators should precisely match with the spectrum of energies for dual string states!*
- Still cannot Quantize Strings in general RR Fluxed-background! Exact Quantization Only in Plane-Wave Limit! (Berenstein-Maldacena-Nastase).
- Semi-classical Quantization can be performed for string states with large quantum numbers!! (Frolov-Tseytlin + Others)

Recap on Spin-Chain and Spinning Strings

- In the planar limit, interested in the single trace, gauge invariant operators, consisting of
$$\left(\Phi_I, \mathcal{D}_\mu, \Psi_\alpha^A\right) .$$
- Physical operators should be of finite lengths, the trace condition then gives the periodicity.
- For example, in $SU(2)$ sector of $\mathcal{N} = 4$ SYM, the typical operator consist of J $Z = \frac{1}{\sqrt{2}}(\Phi_5 + i\Phi_6)$ and Q $X = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$:
$$\mathcal{O}_{SU(2)} \sim \mathrm{Tr}(ZZXZX \ldots ZXZ) + \text{Cyclic Permutation}\,,$$
J and Q can be large but should be finite.
- The problem of understanding the spectrum is to understand their scaling dimensions.
- Can introduce "Dilatation Operator" $\mathfrak{D} = \mathfrak{D}^{(0)} + \delta\mathfrak{D}(\lambda)$ acting on an operator, the eigenvalue is the scaling dimension.

- The anomalous dimension $E(\lambda, J, Q)$ becomes the eigenvalue of anomalous dimension operator $\delta\mathfrak{D}(\lambda)$.
- For example, the classical and one-loop dilatation operators for the $SU(2)$ sector are given by:

$$\begin{aligned} \mathfrak{D}^{(0)} &= \mathrm{Tr}(Z\partial/\partial Z + X\partial/\partial X)\,, \\ \mathfrak{D}^{(1)} &= -\frac{\lambda}{8\pi^2 N}\mathrm{Tr}\,[X, Z]\,[\partial/\partial X, \partial/\partial Z]\ . \end{aligned}$$

- The action of $\mathfrak{D}^{(1)}$ causes Huge Mixing Problem!!

$$\mathfrak{D}^{(1)}\mathrm{Tr}(ZXZZ\ldots) = \#\mathrm{Tr}(ZXZZ\ldots) + \#\mathrm{Tr}(ZZXZ\ldots) + \#\mathrm{Tr}(ZZZX\ldots) + \ldots$$

- Enhancement for $\mathfrak{D}^{(1)}$ appears when $\ldots ZXZZ\ldots$!!

- In the planar limit, map $\delta\mathfrak{D}$ to different Spin Chain Hamiltonians (Minahan and Zarembo; Beisert, Staudacher).
- For example, $SU(2)$ sector, Z "down spin $\downarrow$" and X "up spin $\uparrow$" so that

$$\mathrm{Tr}(ZZXZX\ldots ZXZ) \longrightarrow |\downarrow\downarrow\uparrow\downarrow\uparrow\ldots\downarrow\uparrow\downarrow\rangle \,.$$

- At one loop, $\mathfrak{D}^{(1)}$ becomes Heisenberg "XXX" spin chain Hamiltonian in condensed matter

$$\mathfrak{D}^{(1)} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{L}(\mathbf{1}_{i,i+1} - \mathbf{P}_{i,i+1})\,,$$

$$\mathbf{1}_{i,i+1}|\uparrow\downarrow>=|\uparrow\downarrow>\,, \quad \mathbf{P}_{i,i+1}|\uparrow\downarrow>=|\downarrow\uparrow> \;.$$

- This is an Integrable Hamiltonian.
- Finding $E_1(\lambda) \equiv$ Diagonalization of $\mathfrak{D}^{(1)}$, which can be done by Bethe Ansatz Techniques.

- In thermodynamic limit $J, Q \to \infty$, Q/J fixed, Bethe Equations reduces to integral equations.
- The scaling dimension can then be shown to have expansion in $L = J + Q$

$$\Delta(\lambda) = L + \frac{\lambda}{L}(a_{(1)}^{(0)} + \frac{a_{(1)}^{(1)}}{L} + \ldots) + \frac{\lambda^2}{L^3}(a_{(2)}^{(0)} + \frac{a_{(1)}^{(2)}}{L} + \ldots) + \ldots$$

- Can do the same for other larger sectors and at higher loop orders.

Spinning Strings....

- Consider classical closed string action on $AdS_5 \times S^5$.

$$-\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma (L_{AdS_5} + L_{S^5}) + \text{fermions}\,.$$

- Large J and Q etc. become angular momenta of the string states S^5.
- Cyclicity becomes $X(\sigma) = X(\sigma + 2\pi)$ condition for closed string.
- Spinning String Solutions arise from specific ansatze on worldsheet embeddings.
- The string action reduces to Integrable Models, e.g. Neumann or Neumann-Rosochatius (Frolov, Tseytlin + apologies to many others).
- The energy of the string can be given in terms of Elliptic Functions/Integrals, and also has expansion

$$E(\lambda) = L + \frac{\lambda}{L}(c^{(0)}_{(1)} + \frac{c^{(1)}_{(1)}}{L} + \dots) + \frac{\lambda^2}{L^3}(c^{(0)}_{(2)} + \frac{c^{(2)}_{(1)}}{L} + \dots) + \dots$$

- AdS/CFT demands $\Delta(\lambda) = E(\lambda)$, confirmed at one and two loops.
- At three loops, does not work once we consider leading $\frac{1}{L}$ correction.
- Not a problem of AdS/CFT, but order of limits problem.
- Gauge Theory: λ small, first expand in λ, then $\frac{1}{L}$.
- String Theory: λ large, first expand in $\frac{1}{L}$, then $\frac{\lambda}{L^2}$.
- Different Limiting Procedure should be taken? So the problem simplify?
- Can separate the "finite size effects" of the form $\frac{1}{L}$ and "Stringy effects" of the form $\frac{1}{\lambda}$?

Scattering Matrice in Gauge/String Theories

Such special limit exists (Staudacher; Beisert; Hofman, Maldacena),

$$\begin{aligned} & J \to \infty\,, \quad \Delta \to \infty\,, \\ & \Delta - J \ \text{ fixed}\,, \quad \lambda \ \text{ fixed}\,. \end{aligned} \tag{1}$$

Scattering Matrix for $SU(2)$ sector

- Consider again $SU(2)$ sector, a field X or up spin $\uparrow$ is known as single Magnon.
- Different operators can be classified by different magnon number Q.
- Spectrum is not encoded in Bethe Equations, but the Scattering Matrix.
- Relax the cyclicity condition and consider the case $Q = 1$ with conserved momentum

p (Beisert,2005):

$$|p>=\sum_{l=1}^{L} e^{ipl}|l>\,,\quad |l>\equiv |Z\ldots Z\underbrace{X}_{l-th}Z\ldots Z>\,,$$

- At one loop, energy of $|p>$ is given by the XXX Hamiltonian $\mathfrak{D}^{(1)}$:

$$\mathfrak{D}^{(1)}|p>=4\sin^2\left(\frac{p}{2}\right)|p>\ .$$

- Move on to $Q=2$, with momenta p_1 and p_2, again can write down

$$|p_1,p_2>=\sum_{1\leq l<k\leq L}^{L}\Psi(p_1,p_2)|\downarrow\ldots\underbrace{\uparrow}_{l-th}\downarrow\ldots\downarrow\underbrace{\uparrow}_{k-th}\downarrow\ldots\downarrow>$$

- The natural form of two magnon wave function $\Psi(p_1,p_2)$ is given by

$$\Psi(p_1,p_2)=e^{p_1 l+p_2 k}+\underbrace{\hat{S}(p_1,p_2)}_{S-matrix}e^{p_1 k+p_2 l}\,.$$

- $|p_1, p_2>$ is again an eigenstate of $\mathfrak{D}_1$, with energy

$$E_1(p_1, p_2) = 4\sin^2\left(\frac{p_1}{2}\right) + 4\sin^2\left(\frac{p_2}{2}\right) \; .$$

- Given the energy $E_1(p_1, p_2)$, can also obtain the S-matrix

$$\hat{S}(p_1, p_2) = \frac{u_1 - u_2 + i}{u_1 - u_2 - i}, \qquad \underbrace{u_i}_{rapidity} = \frac{1}{2}\cot\left(\frac{p_i}{2}\right) \; .$$

- Beyond one loop, all loop scattering matrix was proposed in (Beisert,Dippel, Staudacher 2004).
- The form of the scattering matrix remains the same, however the rapidity changes to

$$u_k = \frac{1}{2}\cot\left(\frac{p_k}{2}\right)\sqrt{1 + 8g^2\sin^2\left(\frac{p_k}{2}\right)}, \quad g^2 = \frac{\lambda}{8\pi^2} \, .$$

- $Q > 2$, Factorized Scattering allows us to construct scattering matrix by bootstrap method (illustration later).
- Can introduce alternative parameterizations, the "spectral parameter" x

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 2g^2/u^2}\right) \,, \quad u(x) = x + \frac{g^2}{2x} \,.$$

$$x^{\pm} = x(u \pm i/2) \,, \quad p = -i \log \frac{x^+}{x^-} \,.$$

- This parametrization will be useful for bound states later.

Scattering Matrix for $\mathcal{N}=4$ SYM

- Can generalize to full $\mathcal{N}=4$ SYM, now consider

$$\ldots Z \ldots Z \mathcal{X} Z \ldots Z \cdots + (\text{other insertions}) \, .$$

- The impurity/magnon $\mathcal{X} = (\Phi_I, D_\mu, \Psi^A_\alpha)$ (sixteen flavors) transforms as bi-fundamental representation

$$(\Box;\Box)$$

under the $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \underbrace{\mathbb{R}}_{\Delta - J}$ residual symmetry group of $\mathfrak{psu}(2,2|4)$ of $\mathcal{N}=4$ SYM.

- Residual symmetry preserves the "Vacuum" $\ldots ZZZ \ldots$.
- The bosonic part of the residual algebra contains $SO(4) \times SO(4)$, preserves the killing vector dual to Z in string theory.

- Focus on single $\mathfrak{su}(2|2) = \mathfrak{psu}(2|2) \ltimes \mathbb{R}$ only, the fundamental representation is given by $2+2$-dimensional superspace ("half magnon")

$$(\phi_a, \psi_\alpha)^{\mathrm{t}}\,.$$

- Need two extra central charges, the algebra is extended to $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$.
- The algebra is then given by

$$\begin{aligned}
&[\mathfrak{R}^a{}_b, \mathfrak{J}^c] = \delta^c_b \mathfrak{J}^a - \frac{1}{2}\delta^a_b \mathfrak{J}^c\,, \quad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{J}^\gamma] = \delta^\gamma_\beta \mathfrak{J}^\alpha - \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}^\gamma\,,\\
&\{\mathfrak{Q}^\alpha{}_a, \mathfrak{S}^b{}_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta \mathfrak{C}\,,\\
&\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{P}\,, \qquad \{\dot{\mathfrak{Q}}^{\dot\alpha}{}_{\dot a}, \dot{\mathfrak{Q}}^{\dot\beta}{}_{\dot b}\} = \epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot a\dot b}\mathfrak{P}\,,\\
&\{\mathfrak{S}^a{}_\alpha, \mathfrak{S}^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathfrak{K}\,, \qquad \{\dot{\mathfrak{S}}^{\dot a}{}_{\dot\alpha}, \dot{\mathfrak{S}}^{\dot b}{}_{\dot\beta}\} = \epsilon^{\dot a\dot b}\epsilon_{\dot\alpha\dot\beta}\mathfrak{K}\,.
\end{aligned}$$

- The action of the algebra on the fundamental representation

$$\mathfrak{R}^a{}_b|\phi^c\rangle = \delta_b^c|\phi^a\rangle - \frac{1}{2}\delta_b^a|\phi^c\rangle\,, \quad \mathfrak{L}^\alpha{}_\beta|\psi^\gamma\rangle = \delta_\beta^\gamma|\psi^\alpha\rangle - \frac{1}{2}\delta_\beta^\alpha|\psi^\gamma\rangle\,,$$
$$\mathfrak{Q}^\alpha{}_a|\phi^b\rangle = a\delta_a^b|\psi^\alpha\rangle\,, \quad \mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle = b\epsilon^{\alpha\beta}\epsilon_{ab}|\phi^b Z^+\rangle\,,$$
$$\mathfrak{S}^a{}_\alpha|\phi^b\rangle = c\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\beta Z^-\rangle\,, \quad \mathfrak{S}^a{}_\alpha|\psi^\beta\rangle = d\delta_\alpha^\beta|\phi^a\rangle\,.$$

- Closure of the extended algebra $\mathfrak{psu}(2|2)\times\mathbb{R}^3$ then gives the Exact Magnon Dispersion Relation

$$\Delta - J = \sqrt{1 + 8g^2\sin^2\left(\frac{p}{2}\right)}\,.$$

- It is a BPS state of $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$ and has sixteen-fold degeneracies.

- Can also construct the scattering matrix between two magnons from the algebra

$$\ldots Z \underbrace{\mathcal{X}}_{\overrightarrow{p}} \ldots ZZ \ldots \underbrace{\mathcal{X}'}_{\overleftarrow{p'}} Z \ldots$$

$$\xrightarrow{S(p,p')} \qquad \ldots Z \underbrace{\mathcal{X}''}_{\overleftarrow{p'}} \ldots ZZ \ldots \underbrace{\mathcal{X}'''}_{\overrightarrow{p}} Z \ldots$$

- Integrability needs the momemta p and p' to be conserved, even though magnon flavours may change.
- There are 16^4 states, can simplify by having

$$S(p,p') \cong \underbrace{S(p,p')_{SU(2|2)}}_{16^2 \text{ states}} \times \underbrace{\tilde{S}(p,p')_{SU(2|2)}}_{16^2 \text{ states}}$$

- The S-matrix $S(p_1, p_2)_{SU(2|2)}$ is then constrained by its invariance under the algebra $\mathfrak{psu}(2|2) \times \mathbb{R}^3$.

$$[S(p_1, p_2), \mathfrak{J}_1 + \mathfrak{J}_2] = 0$$

- Also Yang-Baxter equation and Unitarity

$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12} \,, \quad S_{12} S_{21} = I$$

- The residual symmetry allows us to determine the scattering matrix for $\mathcal{N} = 4$ SYM up to an Overall Dressing Factor σ^2.
- The form of the Dressing Factor had been proposed earlier (at lowest order) (Arutyunov, Frolov, Staudacher):

$$\sigma^2(p_k, p_j) = \exp\left(2i \sum_{r=2}^{\infty} \left(\frac{g^2}{2}\right)^r [q_{[r}(p_k) q_{r+1]}(p_j)]\right) \,.$$

- The $q_r(p_k)\,, \quad r = 2\,, \ldots\,, \infty$ are the higher conserved charges carried by the magnon

$$q_r(p_k) = \frac{1}{g^{r-1}} \frac{2\sin\left(\frac{r-1}{2}p_k\right)}{r-1} \left(\frac{\sqrt{1+8g^2\sin^2\left(\frac{p_k}{2}\right)}-1}{2g\sin\left(\frac{p_k}{2}\right)} \right)^{r-1} .$$

- Appeared as an "Interpolating Factor" between "gauge" and "stringy" spin-chains.
- Correctly reproduced the "Near BMN limit" and famous "$\Delta \cong \sqrt[4]{\lambda}$".
- Can interpret $S(p_1, p_2) = \sigma^2 \hat{S}(p_1, p_2)$ as the Stringy Scattering Matrix.

- As a phenomenological application, one can consider the $SL(2)$ sector containing D_- and Z, as it has "Twist Two" operator:

$$Tr(D_-^S Z^2) + \dots \tag{2}$$

its anomalous dimension can simply be given by scattering matrix with dressing phase !!

- Remarkable match with the "Honest Field Theory Calculation" for QCD upto three-loops!! (Moch, Vermaseren, Vogt)
- Further predict the anomalous dimensions for "higher twist operators" and arbitrary D_- insertions to higher loops!!
- Derive magnon dispersion relation from string theory?
- Derive magnon scattering matrix from string theory?
- Studying the asymptotic spectrum of $\mathcal{N} = 4$ SYM, like bound states, breathers?

Magnon Bound states in Gauge Theory

- The spectrum of $\mathcal{N}=4$ SYM in limit (1) should consist of elementary magnons and their infinite tower of bound states.
- Focus on $SU(2)$ sub-sector for simplicity, the all loop S-matrix between two magnons is

$$\hat{S}(p_1,p_2)=\frac{u_1-u_2+i}{u_1-u_2-i}=\frac{x_1^+-x_2^-}{x_1^--x_2^+}\frac{1-g^2/2x_1^+x_2^-}{1-g^2/2x_1^-x_2^+}\,.$$

- The Magnon Bound state appears when simple pole appears in $S(p_1,p_2)$ at complex momenta (Dorey 2006)

$$u_1-u_2=i\,,\quad \text{or}\quad x_1^-=x_2^+\,.$$

- The normalizability of the bound state can be seen at one-loop

$$x_i^{\pm}\approx u_i\pm\frac{i}{2}\,,\quad \Psi(l_1,l_2)\approx\underbrace{\left[\cos\left(p/2\right)\right]^{l_2-l_1}}_{\text{Decaying}}e^{ip/2(l_1+l_2)}\,.$$

- The exact dispersion relation of the bound state is then given by

$$\Delta - J = \sqrt{4 + 8g^2 \sin^2 \frac{p}{2}}\,, \quad p = p_1 + p_2\,.$$

- For general Q magnons, again factorizability gives

$$u_k - u_{k+1} = i\,, \quad \text{or} \quad x_k^- = x_{k+1}^+\,, \quad k = 1, 2, \ldots, Q-1\,.$$

- The exact dispersion relation for Q magnon bound state

$$\Delta - J = \sqrt{Q^2 + 8g^2 \sin^2 \left(\frac{p}{2}\right)}\,.$$

- The exactness of the dispersion again comes from Supersymmetry (HYC,Dorey, Okamura; Beisert)
- The magnon bound states again transform in the short representations of $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)) \ltimes \mathbb{R}^3$

$$(Q - boxes; Q - boxes) \,,$$

where $Q - boxes$ corresponds to super-symmetrized representation.
- The $SU(2)$ bound state given here is one state of the multiplet containing many other flavors.
- $\Delta - J$ is again given by the central charge carried by the multiplet, all flavors share the same dispersion relation!
- Can show that $(Q - boxes; Q - boxes)$ correctly produces the magnon bound state dispersion relation.
- Valid for all λ and Q, reproduce it in string theory for $\lambda, Q \gg 1$?
- Scattering matrix for the magnon bound states?

The Giant and Dyonic Giant Magnons

- To find string states that reproduce magnon and bound state dispersion relations.
- $J \to \infty$ limit, rescaling the worldsheet, string becomes infinitely long (Hofman, Maldacena, 2006).
- The worldsheet now change from a cylinder to a plane.
- The elementary magnon $J \to \infty$, corresponds to σ-model on $R \times S^2$.
- The magnon bound state $J \to \infty, Q \approx \sqrt{\lambda}$, corresponds to σ-model on $R \times S^3$.
- Reduction Procedure to known integrable models (Pohlmeyer, 1976):

$$\begin{aligned} \sigma - \text{model on } R \times S^2 \quad &\to \quad \text{sine Gordon Model} \, , \\ \sigma - \text{model on } R \times S^3 \quad &\to \quad \text{Complex sine Gordon Model} \, . \end{aligned}$$

- Elementary magnons and magnon bound states correspond to classical solitons/kinks in each model.

Brief Notes on Complex sine Gordon Model

- The equation of motion for CsG field ψ

$$\partial_+\partial_-\psi + \psi^* \frac{\partial_+\psi\partial_-\psi}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0\,.$$

- This has exact one soliton solution

$$\psi_{\text{1-soliton}} = e^{i\mu}\frac{\cos(\alpha)\exp(i\sin(\alpha)T)}{\cosh(\cos(\alpha)(X-X_0))}\,.$$

with boosted world sheet coordinates

$$X = \cosh(\theta)x - \sinh(\theta)t\,, \quad T = \cosh(\theta)t - \sinh(\theta)x\,.$$

- The theory has two parameters θ (rapidity) and α (internal $U(1)$ angle).
- Reduces to sine Gordon model when $\alpha = 0$.

The string worldsheet embedding are given by

$$Z_1 = X_1 + iX_2\,, \quad \text{and} \quad Z_2 = X_3 + iX_4\,, \quad |Z_1|^2 + |Z_2|^2 = 1$$

Giant Magnon Solution

$$Z_1 = \left[\sin\left(\frac{p}{2}\right)\tanh(Y) - i\cos\left(\frac{p}{2}\right)\right]\exp(it)\,,$$

$$Z_2 = \frac{\sin\left(\frac{p}{2}\right)}{\cosh(Y)}\,, \quad Y = \frac{x - \cos\left(\frac{p}{2}\right)t}{\sin\left(\frac{p}{2}\right)}\,.$$

(Hofman,Malacena, 2006)

- The two end points correspond to $x = \pm\infty$ on the worldsheet.

- The momentum carried by the magnon is given by $p = \Delta\varphi$ the angle extended by the string in the $S^1 \subset S^5$.
- Correctly reduced to "sine-Gordon Soliton".
- The Giant Magnon has infinite energy Δ and angular momentum J, however the combination

$$\Delta - J = 2\sqrt{2}g\sin\left(\frac{p}{2}\right) \ .$$

precisely reproduce the strong coupling limit of the exact dispersion relation!

Dyonic Giant Magnon Solution

$$
\begin{aligned}
Z_1 &= \frac{1}{\sqrt{1+k^2}} \left(\tanh\left[\cos(\alpha)X\right] - ik\right) \exp(it) \,, \\
Z_2 &= \frac{1}{\sqrt{1+k^2}} \frac{1}{\cosh\left[\cos(\alpha)X\right]} \exp\left(i\sin(\alpha)T\right) \,, \\
k &= \frac{\sinh\theta}{\cos\alpha} = \cot\left(\frac{p}{2}\right) \,.
\end{aligned}
$$

(HYC,Dorey,Okamura; Arutyunov, Frolov, Zamaklar; Minahan, Tirziu, Tseytlin; Spradlin, Volovich)

- The solution now has second non-zero angular momentum Q and its an Action Variable.
- Correctly reduced to Complex sine-Gordon Soliton.
- Correctly gives the dispersion relation for SU(2) magnon bound states.

Scattering of Magnon Bound states

Elementary Magnon Scattering

- At $\lambda \gg 1$, scattering of elementary magnons = scattering of giant magnons.
- Strong coupling limit of magnon scattering matrix, it is dominated by AFS dressing factor

$$\log S(p_j, p_k) \approx -2\sqrt{2}g \left[\cos\left(\frac{p_j}{2}\right) - \cos\left(\frac{p_k}{2}\right)\right] \log\left[\frac{\sin^2\left(\frac{p_j - p_k}{4}\right)}{\sin^2\left(\frac{p_j + p_k}{4}\right)}\right] .$$

- GM=sG soliton, consider the scattering phase of two sG solitons (Hofman, Maldacena, 2006).
- Using the GM solution, sG scattering phase precisely reproduce above! Should be considered as an derivation for AFS dressing factor.

Bound state Scattering (HYC,Dorey,Okamura; Roiban)

- Consider scattering of two magnon bound states of charge Q_1 and Q_2, factorized scattering allows for Bootstrap method.
- The scattering matrix for bound states is

$$\begin{aligned} S_{BS}(X^+, X^-, Y^+, Y^-) &= \exp(i\Theta(X^+, X^-, Y^+, Y^-)) \\ &= \prod_{j_1=1}^{Q_1} \prod_{j_2=1}^{Q_2} S\left(x^+_{j_1}, x^-_{j_1}; y^+_{j_2}, y^-_{j_2}\right) \ , \\ S(x,y) &= \underbrace{\sigma^2(x,y)}_{AFS} \times \underbrace{\hat{S}(x,y)}_{BDS} \ . \end{aligned}$$

- Impose the boundstate conditions $x^-_{j_1} = x^+_{j_1+1}$, etc., can deduce

$$X^+ = x^+_1 \ , \quad X^- = x^-_{Q_1} \ ; \quad Y^+ = y^+_1 \ , \quad Y^- = y^-_{Q_2} \ .$$

- In $g \longrightarrow \infty$ limit, the scattering phase $\Theta(X^+, X^-, Y^+, Y^-)$ becomes

$$\sqrt{2}g[K(X^+, Y^+) + K(X^-, Y^-) - K(X^+, Y^-) - K(X^-, Y^+)]\,.$$

- Here $K(X, Y) = K_0(X, Y) + \hat{K}(X, Y)$, where $K_0(X, Y)$ is from AFS, and $\hat{K}(X, Y)$ is from BDS.
- More specifically, the AFS contribution has the form

$$K_0(X, Y) = -\left[\left(X + \frac{1}{X}\right) - \left(Y + \frac{1}{Y}\right)\right] \log\left(1 - \frac{1}{XY}\right)\ ,$$

- Need to sum over all Q_1, Q_2 magnons, when bound state conditions are imposed, the summation is simplified.
- The BDS contribution $\hat{K}(X, Y)$ is given by

$$\left[\left(X + \frac{1}{X}\right) - \left(Y + \frac{1}{Y}\right)\right] \log\left[(X - Y)\left(1 - \frac{1}{XY}\right)\right]\ .$$

- Derived from approximate the exact product expression by its Riemann Integral.

- Combine both contributions together, $K(X,Y)$ is given by

$$K(X,Y) = [(X + 1/X) - (Y + 1/Y)] \log (X - Y) \ .$$

- Comparison with the time-delay $\Delta\tau = \frac{\partial\Theta}{\partial E}$ of two CsG solitons (Dorey,Hollowood, 1994)
- Use the dictionary:

$$\begin{aligned} \Delta - J &= 2\sqrt{2}g\frac{\cos(\alpha)\cosh(\theta)}{\cos^2(\alpha)+\sinh^2(\theta)} \ , \\ Q &= 2\sqrt{2}g\frac{\cos(\alpha)\sin(\alpha)}{\cos^2(\alpha)+\sinh^2(\theta)} \ . \end{aligned}$$

- Deduce that

$$X^{\pm} = \coth\left(\frac{\theta_1}{2} \pm i\left(\frac{\alpha_1}{2} - \frac{\pi}{4}\right)\right) \ , \quad Y^{\pm} = \coth\left(\frac{\theta_2}{2} \pm i\left(\frac{\alpha_2}{2} - \frac{\pi}{4}\right)\right) \ .$$

- Finally, precise match with the CsG time-delay!
- A non-trivial check of the integrability in the $SU(2)$ subsector.

Conclusion and Work in Progress

- Discussed magnon bound states in gauge/string theories.
- Discussed their classification and scattering.
- Dressing factor receives further λ corrections (Hernandez, Lopez).
- Calculate these from a "first principle" approach? As perturbation from soliton scattering matrix?
- Understanding magnons and bound states as finite-gap solutions?
- Scattering of bound states of different flavors? (Work in Progress with Volovich and Kalosious)
- Understanding the weak coupling interpolation of dressing factor? Transcendentality?