HW#2, Due 9:00am, Sep.30 (Wed).

No late HW will be accepted. So turn in whatever you have done.

1. (a) Show that the sum of two projection operators cannot be a projection operator unless their product is zero. (b) Show that the product of two projection operators cannot be a projection operator unless they commute.

2. Let U be a unitary operator. Consider the eigenvalue equation

$$U|\lambda>=\lambda|\lambda>$$

   (a) Prove that λ is of the form $e^{i\theta}$ with θ real. (b) Show that if $\lambda \neq \mu$ then $<\mu|\lambda>= 0$.

3. A skew-Hermitian operator A is an operator satisfying $A^\dagger = -A$. (a) Prove that A can have at most one real eigenvalues (which may be degenerate) (b)Prove that the commutator of two Hermitian operators is skew-Hermitian.

4. (a) Consider two operators A, B that do not necessarily commute. Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\}$$

   where

$$A^0\{B\} = B,\ A^1\{B\} = [A, B],\ A^2\{B\} = [A, [A, B]], \text{ etc}$$

   (b) Let $A(x)$ be an operator that depends on a continuous parameter x. Derive the following identity

$$e^{-iA(x)} \frac{d}{dx} e^{iA(x)} = i \sum_{n=0}^{\infty} \frac{(-i)^n}{(n+1)!} A^n \left\{ \frac{dA}{dx} \right\}$$

5. Consider the 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   (a) Prove that

$$\exp(i\theta \vec{\sigma} \cdot \hat{n}) = I \cos\theta + i(\vec{\sigma} \cdot \hat{n}) \sin\theta$$

   where $\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$. Here $\hat{n}$ is an arbitrary unit vector, and θ an arbitrary angle.
   (b) Prove that, given any two vectors operators A, B that commute with $\vec{\sigma}$(but not necessarily with each other), we have the identity,

$$(\vec{\sigma} \cdot A)(\vec{\sigma} \cdot B) = A \cdot B + i\vec{\sigma} \cdot (A \times B).$$