Quantum Mechanics-1 Final Exam, Jan.11, 2010.

You must present the derivation or the procedures how you get the answer to get credits.

1. (20%) A particle whose wave function is given by

$$\psi(\vec{r}) = \left(\frac{1}{\sqrt{3}} Y_{11}(\theta, \phi) - \frac{1}{2} Y_{1-1}(\theta, \phi) + \frac{1}{\sqrt{2}} Y_{10}(\theta, \phi) \right) f(r)$$

   where $f(r)$ is a normalized radial function, namely, $\int_0^\infty r^2 f^2(r) dr = 1$.
   (a) Calculate the expectation values of $\hat{L}^2, \hat{L}_z$ and $\hat{L}_x^2$ in this case.
   (b) What is the expectation value of $V(\theta) = 2\cos^2\theta$ in this case.

2. (20%)A perfectly elastic steel sphere of diameter 1 cm is dropped vertically from a height 10 cm above another fixed identical sphere. Using the uncertainty principle to estimate the maximum number of bounces that can occur.

3. (20%) Show that it is impossible for an electron to be in a state such that

$$\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$$

   where $S_i = \frac{1}{2}\hbar\sigma_i$.

4. (20%) (a) Measurement of an electron's spin along the $z-$axis (S_z) using a Stern-Gerlach apparatus gives the eigenvalue $+\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yields $+\hbar/2$?
   (b) Measurement of an electron's spin along the axis $\hat{n}$ gives the eigenvalue $+\hbar/2$. What is the probability that a subsequent measurement of the spin along the $z-$axis yields $-\hbar/2$?

5. (20%) Consider a spinless particle with charge q, and mass m, moving in the constant B field, $\vec{B} = B\{0, 0, 1\}$. Show that the ground-state eigenfunctions with angular momentum $\mu\hbar$ are

$$\psi_{0\mu}(x, y) = \frac{1}{\sqrt{2\pi\mu!}} \left(\frac{x + iy}{\sqrt{2} l_B} \right)^\mu \exp\left(-\frac{x^2 + y^2}{4 l_B^2} \right)$$

   where $l_B = \sqrt{\hbar c / qB}$.

6. For simplicity, let's assume that $m = \hbar = 1$. By means of creation and destruction operators (you actually need two pairs of them),
   (a)(10%) find the energy levels of a two-dimensional harmonic oscillator with potential

$$V = \frac{1}{2}(x^2 + y^2)$$

(b)(10%) Find the degeneracy of each level.
(c)(10%) Write out the wave functions of the ground state and each of the first excited states.
(d)(10%) Identify the latter with eigenstates of angular momentum.

7. (20%) A neutron beam is split into two coherent parts at P, one of which takes path C_1 and the other C_2 on the xy plane. These parts are reunited at Q. Enclosed by C_1 and C_2, there is an infinite long line charge along the z direction, of charge per unit length λ, which generates a radial, cylindrically symmetric electric field E. The Lagrangian for the neutron is

$$L = \frac{1}{2}mv^2 + \mu\vec{\sigma} \cdot \frac{\vec{E} \times \vec{v}}{c}$$

where μ is the neutron spin magnetic moment in Bohr magnetons, and c is the speed of light. Show that the phase shift δ at Q is

$$\delta = \pm\frac{4\pi\lambda\mu}{\hbar c}$$

with $\pm$ for spin up/down. This is called the Aharonov-Casher effect.

8. (20%) A system consists of three independent subsystems with angular momentum J_1, J_2, and J_3 respectively. So that $[J_a^i, J_b^j] = i\hbar\epsilon_{ijk}J_a^k\delta_{a,b}$, where $a, b = 1, 2, 3$ are the subsystem indices and $i, j, k = x, y, z$. We know that the six operators $\{J_1^2, J_{1Z}, J_2^2, J_{2Z}, J_3^2, J_{3Z}\}$ form a complete set of commuting observables. Now, construct another minimum set of complete operators (there should be six) to describe the system partially in terms of the total angular momentum $J = J_1 + J_2 + J_3$. Explain your choice and prove that they commute with each other.