Quantum Mechanics-1 Midterm, Nov.16, 2009.
10:10 -12:00 (No extension)

$*$ You must detail how you arrive the answer to get credits.
$*$ There are more than 100 points in this problem set. I'd suggest that you work out the short or easy problems first.

1. (10%) An electron is in the spin state, in the basis of $|s_{z,\pm}\rangle$,

$$\chi = \frac{1}{3}\begin{pmatrix} 1+2i \\ 2 \end{pmatrix}$$

   If you measure s_x on this electron, what value can you get, and what is the probability of each value? What is $\langle s_x \rangle$?

2. (40%)Consider a system whose state and two observables A and B are given by

$$|\psi(t)\rangle = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

   (a)(10%) Find the eigenvalues and the normalized eigenstates of A and B.
   (b)(10%) What is the probability that a measurement of B at time t yields 1?
   (c)(10%) We perform a measurement where A is measured first and then, immediately afterwards, B is measured. Find the probability of obtaining a value of 0 for A and a value of 1 for B.
   (d)(10%) Which among the sets of operators $\{\hat{A}\}$, $\{\hat{B}\}$, and $\{\hat{A}, \hat{B}\}$ form a complete set of commuting operators? Why?

3. (30%) A particle of mass m lives in the infinite square well with potential

$$V(x) = \begin{cases} 0, \; 0 \le x \le L \\ \infty, \text{ elsewhere} \end{cases}$$

   (a)(10%) Solve the Schrödinger equation and get the normalized eigenfunction and eigen-energy.
   (b) (10%) Now suppose that the particle has the initial wave function

$$\psi(x,0) = \sqrt{\frac{30}{L^5}}x(L-x), \quad (0 \le x \le L)\,.$$

   Outside the well, $\psi(x,0) = 0$. Find $\psi(x,t)$.
   (c) (10%) Find the expectation value of the energy. You might find some of the

following formulae useful:

$$\int dx\, x \sin ax = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$$

$$\int dx\, x^2 \sin ax = \frac{2x}{a^2}\sin ax + \left(\frac{2}{a^3} - \frac{x^2}{a}\right)\cos ax$$

$$\sum_{n=1,3,5..}\frac{1}{n^2} = \frac{\pi^2}{8},\ \sum_{n=1,3,5..}\frac{1}{n^4} = \frac{\pi^4}{96},\ \sum_{n=1,3,5..}\frac{1}{n^6} = \frac{\pi^6}{960}$$

4. (20%) Consider a charged particle on the $x - y$ plane in a constant magnetic field pointing to z direction, $\vec{B} = \{0, 0, B\}$ with the Hamiltonian

$$H = \frac{\Pi_x^2 + \Pi_y^2}{2m},\ \Pi_i = p_i - eA_i,$$

(a)(10%) Show that $[\Pi_x, \Pi_y] = ie\hbar B$.
(b) (10%)Define the operator $a \equiv (\Pi_x + i\Pi_y)/\sqrt{2e\hbar B}$. Show that $[a, a^\dagger] = 1$.

5. (30%)A particle of mass m is allowed to move ONLY along the circle of radius R on a plane, i.e. $x = R\cos\theta$, $y = R\sin\theta$.
(a)(5%) Show that in classical mechanics the Lagrangian is $L = \frac{m}{2}R^2\dot{\theta}^2$. From now on, θ is regarded as the generalized coordinate of the system.
(b)(5%) What are the canonical momentum p_θ and the Hamiltonian in classical mechanics?
(c)(5%) Now we go to quantum mechanics and promote θ and p_θ to operators which satisfy the canonical commutation relation $[\hat{\theta}, \hat{p_\theta}] = i\hbar$. Now the Hamiltonian is also an operator.

Write down the Hesinberg equation of motion, and solve them. (So far no representation is specified yet.)
(d)(5%) Write down the normalized position(θ)-space wave function for the momentum eigenstate $\hat{p_\theta}|k\rangle = k\hbar|k\rangle$, and show that only $k = n \in \mathbb{Z}$ is allowed because of the periodic condition $\psi(\theta + 2\pi) = \psi(\theta)$.
(e)(5%) Write down the Schrödinger equation in the position space, and show that the following continuity equation for the probability holds

$$\frac{\partial\rho}{\partial t} + \frac{\partial j_\theta}{\partial\theta} = 0,\ \rho = |\psi|^2,\ j_\theta = \frac{\hbar}{2imR^2}\left(\psi^*\frac{\partial\psi}{\partial\theta} - \psi\frac{\partial\psi^*}{\partial\theta}\right).$$

(f)(5%) Calculate ρ and j_θ for the state $|n\rangle$ and show that it has a constant flow of probability.