∴ lecture - 1 Quantum Mechanics - 1 ①

14 Sep. 2009

☆ Sketch the course plan and grading

HW + midterm + final

50 20 30

☆ No late HW will be accepted, solution will be post on the due day

☆ Turn in the HW on time even you have not finished.

☆ If any one has study differculty, talk to me within the first week
不然一切分數照算.

☆ 習題一份作業, ⇐ (下週一交)

☆ Historical account for QM ⟸ on an overview of what are expected to be known by taking this graduate level QM.

☆ ~~spell~~ lay out the whole structure of QM

∗ Math tool, linear vector space ⇐ $\boxed{\sim \text{Griffiths}}$

☆ We will go through the math needed to make sure everyone can follow the course.

☆ Math here, can't be completely rigorous.
compromise must be made. at the level we physicists are comfortable -

☆ Here, we will pretend that properties which hold for finite-dim vector space also hold for infinite-dim.
Which is not correct. However, such an approach works without getting one into serious trouble for most practical applications in QM.
(In the part of scattering theory, one has to be careful!)

☆ $\psi(x)$. wave function for a 1-D QM problem.

renormalizable $\Rightarrow \int |\psi(x)|^2 dx < \infty$

WF not renormalizable can't represent physical states.
∵ the total probability must be unity.
(plane wave is not RN, we will deal with it later)

- the space of complex functions which are Renormalizable constitutes a complex vector space.
- $\psi(x)$

I assume that you are familiar with basic notions of vector space, dimension, span, linear dependence & independence, basis, subspace - -

$\{\psi(x)\}$ is an ∞-dim vector space. (Hilbert space)
a complex, inner product vector space
(with additional properties such as completeness and separability - -)

A Hilbert space can be either finite - or ∞ - dim.
(much simpler) Sakurai restricts his use of the term to ∞-dim.

- In momentum space
$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{-\frac{ipx}{\hbar}} \psi(x)$$

- $$\int dx \, |\psi(x)|^2 = \int dp \, |\psi(p)|^2$$

Both
- $\{\psi(x)\}$ and $\{\phi(p)\}$ form a Hilbert space.

- let $\{u_n(x), n=1 \ldots\}$ be an orthonormal basis.
any W.F can be expanded
$$\psi(x) = \sum_n c_n u_n(x)$$
where $$c_n = \int dx \, u_n^*(x) \psi(x)$$

- If $u_n(x)$ are energy eigenfunctions.
$\{c_1, c_2, \ldots\}$ as WF in energy space

$\Rightarrow \int dx \, |\psi(x)|^2 = \sum_n |c_n|^2$ is finite if $\{\psi(x)\}$ forms a Hilbert space

$\Rightarrow \{c_n\}$ of complex numbers is another Hilbert space.

☆ Thus, we have 3 Hilbert spaces:

$\{\psi(x)\}$, $\{\phi(p)\}$, $\{c_n\}$ all assumed to be renormalizable.

☆ These Hilbert spaces are ~~isomorphic~~ isomorphic.

invertible, norm-preserving, linear transformations.

⇒ They all contain the same information.

⇒ none of these can be taken as more fundamental than the others.

⇒ you can do calculation by using any of the 3.

(representation)

☆ You may think $\psi(x)$ or $\phi(p)$ are more fundamental, which is the bias imposed on us through how we learnt QM, as well as the historical development.

Schrödinger WF is not really defined over physical space, but over configuration space. Furthermore, different Hilbert space corresponds to the measurement of diff. physical observable to measure.

☆ It is best to think of the diff WFs are equivalent descriptions of the same physical reality, none of which is more fundamental than the other.

☆ The mathematical formalism of QM exhibits some form invariance or covariance with respect to the set of observables one choses to represent the state of a system.

This view of QM, and the transformation theory ⟺ representation, was worked out in the early days of the Quantum Theory by Dirac, von Neumann, and others.

☆ A representation independent notation:

Dirac's bra and ket.

* A state of a system : ket vector, $| \rangle$

* The point is that the space of kets is another Hilbert space. an abstract space which is isomorphic to $\{\psi\}, \{\phi\}, \{c_i\} \cdots$ but distinct from any of them.

* One can construct a Hilbert space out of the results of physical measurements. no need to start with WF.
* Forget about WF, we will discuss the mathematical properties of ket spaces.

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* A given physical system is associated with a complex vector space. $\mathcal{E}$

For some system, $\mathcal{E}$ is finite-dim (e.g. spin states)
for others, $\mathcal{E}$ is ∞-dim.

* the vectors of $\mathcal{E}$ are called kets, denoted by $| \rangle$.
multiplication by complex # & addition of kets is defined.
c

$$c|\psi\rangle = |\psi\rangle c$$

exist physical principles to associate a definite (pure) state of a physical system with a ray in $\mathcal{E}$

$$|\psi\rangle = c|\psi_0\rangle \qquad \text{a 1-dim vector subspace of } \mathcal{E}$$

for some complex # c. differ by normalization & overall phase.

$\Rightarrow$ a physical state corresponds to the ray (not to any particular ket)

* scalar or inner product is defined.
$\Rightarrow$ dual space. $\mathcal{E}^*$ is the space of complex-valued linear operators which act on $\mathcal{E}$.
* In Dirac's notation, bras $\langle |$

$$\langle \alpha | : \mathcal{E} \to \mathbb{C}$$: map a ket into complex number

e.g. $|\psi\rangle$ is a ket. $\langle\alpha|(|\psi\rangle)$ is a complex #.

The mapping is linear.

$$\langle\alpha|(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\langle\alpha|(|\psi_1\rangle) + c_2\langle\alpha|(|\psi_2\rangle)$$

☆ don't confuse the <u>bra</u> operator with the familiar linear operator used in QM, such as the H, which acts on kets.

the difference

bra : a complex-valued operator

H : ket-valued operator

☆ $(c\langle\alpha|)(|\psi\rangle) = c\langle\alpha|(|\psi\rangle)$

$(\langle\alpha_1| + \langle\alpha_2|)(|\psi\rangle) = \langle\alpha_1|(|\psi\rangle) + \langle\alpha_2|(|\psi\rangle)$

$\Rightarrow$ the set of all bras acting on a given ket space forms a vector space in its own right. $\Rightarrow$ $\mathcal{E}^*$.

If $\mathcal{E}$ is finite-dim $\Rightarrow$ $\mathcal{E}^*$ is also, has the same dimension.

∞ - dim ..

☆ any two vector spaces of the same dimensionality can be placed in 1-1 correspondence. (but no natural one).

☆ However, if $\mathcal{E}$ possesses a metric (or scalar product)

$\Rightarrow$ there is a natural 1-1 between $\mathcal{E}$ & $\mathcal{E}^*$. (like in QM)

☆ A metric or scalar product g on $\mathcal{E}$

$$g : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$$

takes 2 kets to produce a complex #.

linear

$$g(|\psi\rangle, c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1 g(|\psi\rangle, |\phi_1\rangle) + c_2 g(|\psi\rangle, |\phi_2\rangle) \quad \text{1-(A)}$$

$$g(c_1|\psi_1\rangle + c_2|\psi_2\rangle, |\phi\rangle) = c_1^* g(|\psi_1\rangle, |\phi\rangle) + c_2^* g(|\psi_2\rangle, |\phi\rangle) \quad \text{1-(B)}$$

anti linear

- g is symetric (Hermitian)

$$g(|\psi\rangle, |\phi\rangle) = g(|\phi\rangle, |\psi\rangle)^*$$

- g is positive defined

$$g(|\psi\rangle, |\psi\rangle) \geq 0$$

$= 0$ iff $|\psi\rangle = 0$

by Hermiticity, is real

- Dual correspondence (DC) can be defined given the metric g

$$DC : \mathcal{E} \to \mathcal{E}^*$$

- If $|\psi\rangle$ is a ket, we denote the bra corresponding to it under the DC by $\langle\psi|$. (with the same label symbol)

We define $\langle\psi|$ in terms of its action on an arbitrary ket $|\phi\rangle$

$$\langle\psi|(|\phi\rangle) = g(|\psi\rangle, |\phi\rangle)$$

The scalar product, ($\langle\psi|$ as fixed, $(|\phi\rangle)$ as variable)

a complex valued linear function of kets. (according to 1-(A))

bra

Denote the bra resulting from a given ket under D.C. by a dagger $(\dagger)$

So that, if $DC: |\psi\rangle \to \langle\psi|$

we will write $\langle\psi| = (|\psi\rangle)^\dagger$

We will say that $\langle\psi|$ is the Hermitian conjugate of $|\psi\rangle$.

Note that since the scalar product is anti linear in its first operand, the DC is antilinear :

$$(c_1|\psi_1\rangle + c_2|\psi_2\rangle)^\dagger = c_1^*\langle\psi_1| + c_2^*\langle\psi_2|$$

If $\mathcal{E}$ is finite-dim, it's easy to see that DC is 1 to 1, so ~~that~~ every bra is in fact the Hermitian conjugate of some ket. (for ∞-dim, more restrictions are needed to make it 1-1)

$\Rightarrow$ DC has an inverse $|\psi\rangle = (\langle\psi|)^\dagger$

So that $((|\psi\rangle)^\dagger)^\dagger = |\psi\rangle$

From now on, simplify the notation of scalar product

$$\langle\psi|(|\phi\rangle) = g(|\psi\rangle, |\phi\rangle) = \langle\psi|\phi\rangle$$

$$\Rightarrow \langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \quad \text{and} \quad \langle\psi|\psi\rangle \geq 0$$

$=0$ iff $|\psi\rangle = 0$. called inner product of $|\psi\rangle$ and $|\phi\rangle$.

Schwarz inequality

$$|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$$

with equality iff $|\psi\rangle$ and $|\phi\rangle$ lie in the same ray proof:

$$|\alpha\rangle = |\psi\rangle + \lambda|\phi\rangle \qquad \lambda: \text{complex } \#$$

$$\langle\alpha|\alpha\rangle = \langle\psi|\psi\rangle + \lambda\langle\psi|\phi\rangle + \lambda^*\langle\phi|\psi\rangle + |\lambda|^2\langle\phi|\phi\rangle \geq 0$$

$|\phi\rangle = 0$, Schwarz inequality holds,
consider $|\phi\rangle \neq 0$, and pick $\lambda = -\dfrac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$

$$\Rightarrow \underbrace{\langle\psi|\psi\rangle - \frac{|\langle\psi|\phi\rangle|^2}{\langle\phi|\phi\rangle}}_{\checkmark} \geq 0 \quad \Rightarrow \quad \langle\psi|\psi\rangle\langle\phi|\phi\rangle \geq |\langle\psi|\phi\rangle|^2$$

If it is zero, $\langle\alpha|\alpha\rangle = 0$ implies $|\alpha\rangle = 0$ or $\underbrace{|\psi\rangle = -\lambda|\phi\rangle}_{\text{linear dependent or in the same ray}}$

☆ Operators $\begin{cases} \text{linear} & : L \\ \text{antilinear} & : A \end{cases}$ map from ket space into itself

$$L: \mathcal{E} \to \mathcal{E}, \qquad A: \mathcal{E} \to \mathcal{E}$$

but $\begin{cases} L(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1 L|\psi_1\rangle + c_2 L|\psi_2\rangle \\ A(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1^* A|\psi_1\rangle + c_2^* A|\psi_2\rangle \end{cases}$ only $\mathbb{C}$ (time-reversal) discuss later.

☆ Linear operators can be multiplied by complex # and added
⇒ form a complex vector space in their own right.

☆ Linear operators can also be multiplied.
The multiplication is associative but not in general communitative

$$[A, B] \equiv AB - BA \qquad \text{commutator}$$
$$\{A, B\} \equiv AB + BA \qquad \text{anti commutator}$$

if A possess an inverse A^{-1}
$$\Rightarrow AA^{-1} = A^{-1}A = \mathbb{1}$$