Review
Postulates of QM

① Hilbert space $\mathcal{E}$
② state $\simeq$ ray in $\mathcal{E}$
③ measurement ~ complete Hermitian operator A

④ Exp result ~ $a_1, \cdots a_n$ or $a(\nu)$ of A
⑤ probability $\quad \text{Prob}(A = a_n) = \dfrac{\langle \psi | P_n | \psi \rangle}{\langle \psi | \psi \rangle}$

$\text{Prob}(A \in I) = \dfrac{\langle \psi | P_I | \psi \rangle}{\langle \psi | \psi \rangle}$

⑥ collapse after measurement $\quad \underbrace{P_n \overbrace{|\psi\rangle}^{\text{before}}}_{\text{after}}$

Questions:

What $\mathcal{E}$ is associated with a given physical system?
Which ray corresponds to a definite state of system?
What A is associated with a measurement?

Learn from an example. Stern - Gerlach experiment.

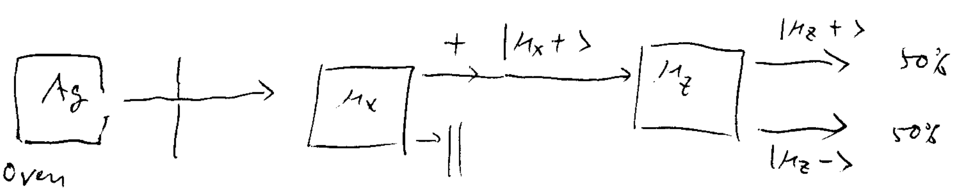
From Modern perspective: Silver atom has an unpaired electron.

$Ag \quad Z = 47 \quad (\underset{\text{krypton}}{\text{Kr}})\, 4d^{10}\, 5s \qquad {}^{2S+1}L_J = {}^2S_{\frac{1}{2}}$

$\Rightarrow$ magnetic moment $= \pm \mu_0$, $\quad \mu_0 = \dfrac{e\hbar}{2m_e c}$ Bohr magneton

$|\vec{\mu}| \propto \text{spin} = \pm \frac{\hbar}{2}$, $\quad \vec{\mu} = \left(\frac{e}{m_e c}\right) \vec{S}$. (Dirac g-factor = 2.)

But we will pretend that we know nothing about it, and talk about magnetic moment, not spin, only.



Since μ_x gives two possible outcomes, according to P③, P④ that μ_x has 2 eigenvalues: $\pm\mu_0$ (also true for μ_y, μ_z)

$\Rightarrow$ $\mathcal{E}$ must be a least 2-dim, since the eigenspace for $\pm\mu_0$ are orthogonal.

For simplicity, we assume they are not degenerate (more on this later!)

Under this assumption, the eigenspaces are 1-Dim,
the ket space $\mathcal{E}$ has exactly 2-dim, it is spanned by
the eigenkets of μ_x with eigenvalues $\pm\mu_0$.

But since it is only 2-dim, it can also be spanned by
the eigenkets of μ_y or μ_z !!
(with eigenvalues $\pm\mu_0$)

$\Rightarrow$ the pair of eigenkets of any of μ_x, μ_y, or μ_z must be expressible as
linear combinations of the eigenkets of any other operator.

denote some normalized eigenvector (up to a phase) of μ_x : $\begin{cases} |\mu_x, +\rangle \\ |\mu_x, -\rangle \end{cases}$

and $|\mu_y \pm\rangle$, $|\mu_z \pm\rangle$

$$\langle \mu_x + | \mu_x - \rangle = \langle \mu_y + | \mu_y - \rangle = \langle \mu_z + | \mu_z - \rangle = 0$$

By P-②, the state of Ag at various stages ~ some state vector

By P-④, after μ_x with $+\mu_0$, the state is $|\mu_x +\rangle$

the state entering the 2nd magnet is a linear combination of $|\mu_z \pm\rangle$

$$\Rightarrow \quad |\mu_x +\rangle = c_+ |\mu_z +\rangle + c_- |\mu_z -\rangle,$$

and $$c_\pm = \langle \mu_z \pm | \mu_x + \rangle$$

By P-⑤ $$\text{Prob}(\mu_z = +\mu_0) = \langle \mu_{x+} | \mathbb{P}_{z+} | \mu_x + \rangle, \quad \mathbb{P}_{z+} = |\mu_z +\rangle\langle \mu_z +|$$
$$= |\langle \mu_{x+} | \mu_{z+}\rangle|^2 = |c_+|^2 = \frac{1}{2}$$

$$\Rightarrow \quad c_+ = \frac{1}{\sqrt{2}} e^{i\alpha_+}, \text{ also } c_- = \frac{1}{\sqrt{2}} e^{i\alpha_-}$$

therefore $$|\mu_x +\rangle = \frac{e^{i\alpha_+}}{\sqrt{2}} \left(|\mu_z +\rangle + e^{i\delta_+} |\mu_z -\rangle \right), \quad (\delta_+ = \alpha_- - \alpha_+)$$

The overall phase δ_+ can be absorbed by redefination of the kets -

☆ Similarly,

$$|\mu_x -\rangle = \frac{1}{\sqrt{2}}\left(|\mu_z +\rangle + e^{i\delta_-}|\mu_z -\rangle\right)$$ δ_- is another phase

by orthogonality

$$\langle \mu_x + | \mu_x -\rangle = \frac{1}{2}\left(1 + e^{i(\delta_- - \delta_+)}\right] = 0$$

$$\Rightarrow \quad |\mu_x \pm\rangle = \frac{1}{\sqrt{2}}\left(|\mu_z +\rangle \pm e^{i\delta_+}|\mu_z -\rangle\right)$$

☆ Similarly, If we first measure μ_y then μ_z, we will arrive at

$$\Rightarrow \quad |\mu_y \pm\rangle = \frac{1}{\sqrt{2}}\left(|\mu_z +\rangle \pm e^{i\theta}|\mu_z -\rangle\right),$$ θ is another phase.

(a note this is Gedanken experiment, μ_y is hard to measured in practice)

☆ Now, we can expand the operators by their projection operator and eigenvales:

$$\mu_x = \mu_0\left(|\mu_x +\rangle\langle \mu_x +| - |\mu_x -\rangle\langle \mu_x -|\right)$$

$$= \mu_0\Big[\tfrac{1}{2}\left(|\mu_z +\rangle + e^{i\delta_+}|\mu_z -\rangle\right)\left(\langle \mu_z +| + e^{-i\delta_+}\langle \mu_z -|\right)$$

$$- \tfrac{1}{2}\left(|\mu_z +\rangle - e^{i\delta_+}|\mu_z -\rangle\right)\left(\langle \mu_z +| - e^{-i\delta_+}\langle \mu_z -|\right)\Big]$$

$$= \mu_0\left[e^{-i\delta_+}|\mu_z +\rangle\langle \mu_z -| + e^{+i\delta_+}|\mu_z -\rangle\langle \mu_z +|\right]$$

similarly, operator μ_y can be expressed in term of $|\mu_z \pm\rangle$ as

$$\mu_y = \mu_0\left[e^{-i\theta}\left(|\mu_z +\rangle\langle \mu_z -|\right) + e^{+i\theta}\left(|\mu_z -\rangle\langle \mu_z +|\right)\right]$$

and the μ_z

$$\mu_z = \mu_0 \left(|\mu_z +\rangle\langle \mu_z -| \quad - \quad |\mu_z -\rangle\langle \mu_z -| \right)$$

* Now imagine that we do μ_x first then μ_y again the outcome shall be 50% $\mu_y = \pm \mu_0$

therefore $\frac{1}{2} = \left| \langle \mu_x + | \mu_y \pm \rangle \right|^2$

$$|\mu_x +\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z +\rangle + e^{i\delta_+} |\mu_z -\rangle \right)$$

$$|\mu_y \pm\rangle = \frac{1}{\sqrt{2}} \left(|\mu_z +\rangle \pm e^{i\theta} |\mu_z -\rangle \right)$$

$$= \left| \frac{1}{2} \pm \frac{1}{2} e^{i(\delta_+ + \theta)} \right|^2$$

$$= \left| \frac{1}{2} \left(1 \pm \cos(\theta - \delta_+) \pm i \sin(\theta - \delta_+) \right) \right|^2$$

$$= \frac{1}{4} \left((1 \pm \cos(\theta - \delta_+))^2 + \sin^2(\theta - \delta_+) \right)$$

$$= \frac{1}{2} \left(1 \pm \cos(\theta - \delta_+) \right) \quad \Rightarrow \quad \theta = \delta_+ \pm \frac{\pi}{2}$$

or $e^{i\theta} = \pm i e^{i\delta_+}$ $\Rightarrow$ only 1 unknown phase δ_+ and 1 unknown sign left.

* The phase δ_+ is now convention, depends on either μ_x or μ_y is purely real in the $|\mu_z \pm\rangle$ basis.
* Traditionally, one chooses μ_x to be real - then ($\delta_+ = 0$ & $\theta = \frac{\pi}{2}$)

$$\mu_x = \mu_0 \left[\left(|\mu_z +\rangle \langle \mu_z -| \right) + \left(|\mu_z -\rangle\langle \mu_z +| \right) \right]$$

$$\mu_y = \pm \mu_0 \left[-i \left(|\mu_z +\rangle \langle \mu_z -| \right) + i \left(|\mu_z -\rangle\langle \mu_z +| \right) \right]$$

$$\mu_z = \mu_0 \left[\left(|\mu_z +\rangle\langle \mu_z +| \right) - \left(|\mu_z -\rangle\langle \mu_z -| \right) \right]$$

☆ In the matrix representation in the $|\mu_z \pm\rangle$ basis :

$$\mu_x \sim \mu_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mu_y \sim \pm\mu_0 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mu_z \sim \mu_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

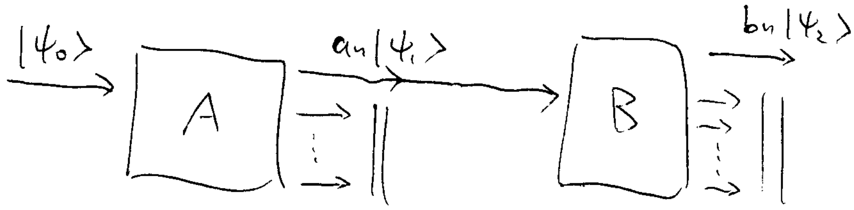
$\Rightarrow$ we rediscover the pauli matrices.
by ~~purely~~ solely using the QM postulates,
and the experimental result !!

Now, the question of degeneracies.

☆ This is not a mathematical problem, we must build up the Hilbert space by the results of physical measurements. —
How do we know wether the eigenspace of μ_x (and μ_y, μ_z) is nondegenerate or not ?

☆ The resolution will rely on the compatible or commuting operators :

Consider the following exp set up, a generalized Stern-Gerlach



The first filter selects the state with eigenvalue a_n, after A,
and the 2nd filter pick the one with $B = b_m$

* According to P-5

$$\text{Prob}(a_n) = \frac{\langle \psi_0 | \mathbb{P}_{A_n} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}$$

$\mathbb{P}_{A_n}$: projection operator onto the eigenspace of A corresponding to eigenvalue a_n

also $\quad |\psi_1\rangle = \mathbb{P}_{A_n} |\psi_0\rangle$

* Next we compute the probability of first $A = a_n$ then $B = b_m$.

$$\Rightarrow \text{Prob}(a_n; b_m) = \frac{\langle \psi_1 | \mathbb{P}_{B_m} | \psi_1 \rangle}{\langle \psi_1 | \psi_1 \rangle} \frac{\langle \psi_0 | \mathbb{P}_{A_n} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}$$

$\mathbb{P}_{B_m}$: projection operator onto the eigenspace of B with eigenvalue b_m.

$$\langle \psi_1 | \psi_1 \rangle = \left(\langle \psi_0 | \underbrace{\mathbb{P}_{A_n}^{\dagger}}_{\text{Hermitian}} \right) \left(\mathbb{P}_{A_n} | \psi_0 \rangle \right) = \langle \psi_0 | \mathbb{P}_{A_n}^2 | \psi_0 \rangle = \langle \psi_0 | \mathbb{P}_{A_n} | \psi_0 \rangle$$

$$= \frac{\langle \psi_1 | \mathbb{P}_{B_m} | \psi_1 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{\langle \psi_0 | \mathbb{P}_{A_n} \mathbb{P}_{B_m} \mathbb{P}_{A_n} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}$$

And we can reverse the ordering, first $B = b_m$ then $A = a_n$

$$\Rightarrow \text{Prob}(b_m; a_n) = \cancel{\frac{\langle \psi_0 | \mathbb{P}_{B_m} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}}$$

$$= \frac{\langle \psi_0 | \mathbb{P}_{B_m} \mathbb{P}_{A_n} \mathbb{P}_{B_m} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}$$

In general, $\quad \text{Prob}(a_n; b_m) \neq \text{Prob}(b_m; a_n)$

· However, if $[A, B] = 0$ then $[\mathbb{P}_{A_n}, \mathbb{P}_{B_m}] = 0$

$$\Rightarrow \mathbb{P}_{A_n}\mathbb{P}_{B_m}\mathbb{P}_{A_n} = \mathbb{P}_{A_n}^2\mathbb{P}_{B_m} = \mathbb{P}_{A_n}\mathbb{P}_{B_m}^2 = \mathbb{P}_{B_m}\mathbb{P}_{A_n}\mathbb{P}_{B_m}$$

$$\Rightarrow \text{Prob}(a_n; b_m) = \text{Prob}(b_m; a_n) = \frac{\langle\psi_0|\mathbb{P}_{A_n}\mathbb{P}_{B_m}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}$$

$\Rightarrow$ reversely, if the prob are equal for all initial state $|\psi_0\rangle$

$\Rightarrow [\mathbb{P}_{A_n}, \mathbb{P}_{B_m}] = 0$;

· And, if this commutator vanishes for all n and m, $[A, B] = 0$

$\Rightarrow$ probability is independent of the order of measurements if and only if the observables commute. (This is the physical meaning of commuting observables)

Question: How do we know if the eigenvalues of the operator A are degenerat

· If a_n is degenerate, $\mathcal{E}_n$ will be multi-dim.
an equivalent question is: what is the dimensionality of $\mathcal{E}_n$?

· The answer is finding an operator B which commutes with A. and see if it can resolve the degeneracy of a_n. (more out comes)

Then, after $A \to B$, the state lies in the simultaneous eigenspace of A & B. or $\mathbb{P}_{A_n}\mathbb{P}_{B_m}|\psi_0\rangle$

So the order of degeneracy of $a_n \geq$ # of outcomes of B-measurem

· But B could also be degenerate? find another operator C ---

$\Rightarrow (A, B, C \text{ ----- })$ until no more degeneraty —

$\Rightarrow$ This set of operators is called [a complete set of commuting observables].

<u>CSCO.</u>

☆ Usually, we do not go all the way to a complete set of commuting obersevables, because we are only interested in some subset of D.O.F of a system.

e.g. In the Stern-~~Ger~~Gerlach experiment,

we ignored the spatial D.O.F, the internal D.O.F

quark, atom, nucleon ...

Say if the spatial D.O.F s are included

the Hilbert space is $2 \otimes \infty$

☆ Now the statistical nature of QM.

$$f_n \equiv \text{Prob}(A = a_n) = \langle \psi | \mathbb{P}_n | \psi \rangle$$

so the average value of a will be

$$\langle a \rangle = \sum_n f_n a_n = \langle \psi | \sum_n a_n \mathbb{P}_n | \psi \rangle = \langle \psi | A | \psi \rangle$$

or we write $\langle a \rangle = \langle A \rangle$

similarly, the standard deviation (variance) Δa^2 can be expressed in terms of Hilbert space operations.

$$\Delta a^2 = \sum_n f_n a_n^2 - \left(\sum_n f_n a_n \right)^2 \quad \text{or} = \sum_n f_n (a_n - \langle a \rangle)^2$$

$$= \sum_n \left[f_n a_n^2 - 2 f_n a_n \langle a \rangle + f_n \langle a \rangle^2 \right]$$

$$= \left(\sum_n f_n a_n^2 \right) - \langle a \rangle^2$$

$$\equiv \langle \psi | \sum_n a_n^2 \mathbb{P}_n | \psi \rangle - \left(\langle \psi | \sum_n a_n \mathbb{P}_n | \psi \rangle \right)^2$$

$$= \langle \psi | A_1^2 | \psi \rangle \quad \text{where } A_1 \equiv A - \langle A \rangle$$

which can be easily shown because $\sum_n a_n^2 \mathbb{P}_n = \left(\sum_n a_n \mathbb{P}_n \right) \left(\sum_m a_m \mathbb{P}_m \right)$

We write $\Delta A^2 = \Delta a^2$

☆ Immediatly, we knows that $\Delta a^2 = 0$ (single values experiment, with 100% prob)

$$\Rightarrow \quad \Delta a^2 = 0 = \langle \psi | A_1^{+} A_1 | \psi \rangle = \langle \phi | \phi \rangle, \quad |\phi\rangle \equiv A_1 |\psi\rangle$$

This implies that $|\phi\rangle = 0$, or $A|\psi\rangle = \langle A \rangle |\psi\rangle$

☆ $\Rightarrow$ A quantum measurement of an observable A with no dispersion iff $|\psi\rangle$ is an eigenstate of A !!

☆ A related subject : generalized uncertainty principle.

$$\boxed{\Delta A^2 \, \Delta B^2 \geq \frac{1}{4} \left| \langle | [A, B] | \rangle \right|^2}$$

proof: $|\alpha\rangle = A_1 |\psi\rangle$, $|\beta\rangle = B_1 |\psi\rangle$

$A_1 = A - \langle A \rangle$, $B_1 = B - \langle B \rangle$

by using the Schwarz inequality, we know

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$$

and

$$\langle \alpha | \beta \rangle = \langle \psi | A_1 B_1 | \psi \rangle$$

$$= \langle \psi | \frac{1}{2} \underbrace{[A_1, B_1]}_{\substack{\text{anti hermitian} \\ \text{purely imaginary}}} | \psi \rangle + \langle \psi | \frac{1}{2} \underbrace{\{A_1, B_1\}}_{\substack{\text{Hermitian} \\ \text{purely real}}} | \psi \rangle$$

Since A_1, B_1 are Hermitian,

therefore

$$|\langle \alpha | \beta \rangle|^2 \geq \frac{1}{4} \left| \langle \psi | [A_1, B_1] | \psi \rangle \right|^2$$

since $\langle A \rangle$ and $\langle B \rangle$ are c-numbers, $[A_1, B_1] = [A, B]$

$$\Rightarrow \quad \Delta A^2 \, \Delta B^2 \geq \frac{1}{4} | [A, B] |^2$$

☆ In the case of $A = x$, $B = p$, $[A, B] = i\hbar$

$$\Rightarrow \quad \boxed{\Delta x \, \Delta p \geq \frac{\hbar}{2}}$$