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* Momentum in CM $\begin{cases} \text{kinetic momentum} & \vec{P} = m\dot{\vec{x}} \\ \text{canonical momentum} & \vec{P} = \dfrac{\partial L}{\partial \dot{\vec{x}}} \end{cases}$

L: Lagrangian,

* In EM background.

$$L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - g\phi(x) + \frac{g}{c}\dot{\vec{x}}\cdot\vec{A}(x)$$

canonical: $\vec{P} = \dfrac{\partial L}{\partial \dot{x}} = m\dot{\vec{x}} + \dfrac{g}{c}\vec{A}(x) \neq \vec{P}_{\text{kinetic}}$

* Question: which corresponds to the momentum operator in QM?

Ans: The canonical one:

reasons: * Hamilton's equation, $H(p,q,t) = \Sigma p\dot{q} - L$

It's the canonical one in $\dot{q} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial q}$

* By Noether's theorem. (symmetry $\rightarrow$ conserved charge)

$L(q_i, \dot{q}_i)$ $\quad \vec{q} \to \vec{q} + \epsilon\vec{n}$, $\delta\vec{q} = \epsilon\vec{n}$

$\delta\dot{\vec{q}} = \dfrac{d}{dt}(\delta\vec{q})$

L invariant $\Rightarrow$ $0 = \delta L = \dfrac{\partial L}{\partial q_i}\delta q_i + \dfrac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i$

$$= \underline{\frac{\partial L}{\partial q_i}\delta q_i} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q_i\right) - \underline{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}\delta q_i$$

by E.O.M $\quad = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\delta q_i\right)$

$\Rightarrow$ $\boxed{\dfrac{\partial L}{\partial \dot{q}_i}\delta q_i \text{ is conserved}}$

or $\vec{P}\cdot\vec{n}$ is conserved. $\Rightarrow$ if $\vec{n}$ isotropic all $\vec{P}$ are const.

$\Rightarrow$ Here, it is the canonical momentum again

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* In CM, momentum is regarded as the generator of translation.

e.g. $f(x, p)$, some function of x & p.

define a classical translation operator $T_{cl}(a)$, such that

$(T_{cl}(a) f)(x, p) = f(x-a, p)$, moves $f(x,p)$ forward in x leaves p intact.

If a infinitesimal

$$(T_{cl}(a) f)(x, p) \simeq f(x, p) - a \cdot \frac{\partial f}{\partial x} + O(a^2)$$

This term can be expressed in terms of Poisson bracket

$$-\vec{a} \cdot \frac{\partial f}{\partial \vec{x}} = \{ \vec{a} \cdot \vec{p}, f \}_{\text{poisson}}$$

according to Sakurai's convention

$$\{ A, B \}_{\text{poisson}} \equiv -\sum_i \left(\frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right)$$

let's check

$$\{ \vec{a} \cdot \vec{p}, f \}_{\text{poisson}} = -\sum_i \left(\frac{\partial (a_k p_k)}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial (a_k p_k)}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

$$= -\sum_i a_i \frac{\partial f}{\partial q_i} = -\vec{a} \cdot \vec{\nabla} f$$

$\Rightarrow$ move function f by An infinitesimal displacement $\vec{a}$ = ~~acting~~ pairing up $(\vec{a} \cdot \vec{p})$ and f with the poisson bracket

$\Rightarrow$ finite displacement can be built up by ∞ infinitesimal ones.

$$\vec{\epsilon} = \frac{\vec{a}}{N} \qquad T_{cl}(a) f = \lim_{N \to \infty} \left(\mathbb{1} + \{ \frac{\vec{a} \cdot \vec{p}}{N}, \cdot \}_{\text{poisson}} \right)^N f$$

$$= \boxed{e^{:\vec{a} \cdot \vec{p}:}} f$$

$$:A: B \equiv \{ A, B \}_{\text{poisson}}$$

$\Rightarrow$ $\boxed{\text{The generator is the canonical momentum !!}}$

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※ Maybe I will spend 1-hr to quickly review the CM.

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We conclude that

* The momentum in QM should correspond to the canonical momentum in CM, not the kinetic.
* QM should respect the salient symmetry $\Leftrightarrow$ conserved quantities (e.g. momentum is conserved for free particle $\Leftrightarrow$ invariant under translation)

Generator of Translations (in QM) $\qquad T(a)|x\rangle = |x+a\rangle$

~~Early~~ Taylor expansion

$$T(\vec{a}) = \mathbb{1} + \vec{a}\cdot\frac{\partial T}{\partial \vec{a}}(0) + \cdots$$

$\hookrightarrow T(0) = \mathbb{1}$

define $\hat{k} = i\frac{\partial T}{\partial \vec{a}}(0)$ (a vector operator)

Then $T(\vec{a}) = 1 - i\vec{a}\cdot\hat{\vec{k}} + \cdots$ (Notice that $\vec{k}$ is independent of $\vec{a}$)

The "i" makes $\hat{k}$ Hermitian, because

$$T(a)\,T(a)^{+} = (\mathbb{1} - i\vec{a}\cdot\vec{\hat{k}} + \cdots)(\mathbb{1} + i\vec{a}\cdot\vec{\hat{k}}^{+} + \cdots)$$
$$= \mathbb{1} - i\vec{a}\cdot(\hat{k} - (\hat{k})^{+}) + \cdots = \mathbb{1}$$

$\Rightarrow \hat{k} = (\hat{k})^{+} \Rightarrow$ it corresponds to some physical vector measurement

---

write $\vec{a} = s\vec{n}$, $\vec{n}$: a unit vector

$$\frac{d}{ds}T(s\vec{n}) = \lim_{\epsilon\to 0}\frac{T((s+\epsilon)\vec{n}) - T(s\vec{n})}{\epsilon} = \lim_{\epsilon\to 0}\frac{T(s\vec{n})\,T(\epsilon\vec{n}) - T(s\vec{n})}{\epsilon}$$
$$= \left(\lim_{\epsilon\to 0}\frac{T(\epsilon\vec{n}) - \mathbb{1}}{\epsilon}\right) T(s\vec{n})$$

from above

& $\quad T(\epsilon\vec{n}) = \mathbb{1} - i\epsilon(\vec{n}\cdot\vec{\hat{k}}) + O(\epsilon^2)$

$$\Rightarrow \frac{d}{ds}T(s\vec{n}) = -i(\vec{n}\cdot\vec{\hat{k}})\,T(s\vec{n}) \Rightarrow$$

first order D.E. with B.C. $T(0) = \mathbb{1}$

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$$\Rightarrow \quad T(s\vec{n}) = \exp(-i\, s\, \vec{n}\cdot\vec{\hat{k}}) \qquad \vec{\hat{k}} = (\hat{k}_1, \hat{k}_2, \hat{k}_3)$$

or $T(\vec{a}) = \exp(-i\,\vec{a}\cdot\vec{\hat{k}})$

$$= \mathbb{1} - i(\vec{a}\cdot\vec{\hat{k}}) - \frac{1}{2}(\vec{a}\cdot\vec{\hat{k}})^2 + \cdots$$

This can be compared to the classical one:

Expand the product $T(a)\,T(b)$

$$T(a)\,T(b) = \left[\mathbb{1} - i(\vec{a}\cdot\vec{\hat{k}}) - \frac{1}{2}(\vec{a}\cdot\vec{\hat{k}})^2 + \cdots\right]\left[\mathbb{1} - i(\vec{b}\cdot\vec{\hat{k}}) - \frac{1}{2}(\vec{b}\cdot\vec{\hat{k}})^2 + \cdots\right]$$

$$= \mathbb{1} - i(\vec{a}+\vec{b})\cdot\vec{\hat{k}} - \frac{1}{2}\left[(\vec{a}\cdot\vec{\hat{k}})^2 + 2(\vec{a}\cdot\vec{\hat{k}})(\vec{b}\cdot\vec{\hat{k}}) + (\vec{b}\cdot\vec{\hat{k}})^2\right] + \cdots$$

but $T(a)\,T(b) = T(b)\,T(a)$, therefore $[\vec{a}\cdot\vec{\hat{k}},\ \vec{b}\cdot\vec{\hat{k}}] = 0$

for arbitrary $\vec{a}$ & $\vec{b}$

$$\Rightarrow \quad [\hat{k}_i, \hat{k}_j] = 0$$

$\vec{\hat{k}}$ is a vector operator.
commuting and Hermitian

☆ Check the commutator, $[\hat{X}_i, T(\vec{a})] = a_i T(\vec{a})$

$$\hat{x}_i T(\vec{a}) = \int d^3\vec{x}\ \ x_i\, T(\vec{a})\,|\vec{x}\rangle\langle\vec{x}| = \int d^3\vec{x}\ \ \hat{x}_i\,|\vec{x}+\vec{a}\rangle\langle\vec{x}|$$

$$= \int d^3\vec{x}\ \ (x_i + a_i)\,|\vec{x}+\vec{a}\rangle\langle\vec{x}|$$

$$= \int d^3\vec{x}\ \ T(\vec{a})\,|\vec{x}\rangle\langle\vec{x}|\,(\hat{x}_i + a_i) \quad \text{c\#} \quad = T(a)\,\hat{x}_i + a_i T(\vec{a})$$

☆ Expand $[\hat{X}_i, T(\vec{a})] = a_i T(\vec{a})$ in $\vec{a}$

$$\hat{x}_i(\mathbb{1} - i\vec{a}\cdot\vec{\hat{k}} + \cdots) - (\mathbb{1} - i\vec{a}\cdot\vec{\hat{k}} + \cdots)\,\hat{x}_i = a_i(\mathbb{1} - i\vec{a}\cdot\vec{\hat{k}} + \cdots)$$

To the leading order $[\hat{x}_i, \vec{a}\cdot\vec{\hat{k}}] = i a_i \Rightarrow \boxed{[\hat{x}_i, \hat{k}_j] = i\delta_{ij}}$

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* ~~The~~ Momentum in QM.

$\hat{k}$: generator of displacement,
- it's Hermitian, (physical)
- it's vector operator ($\vec{x}$, $\vec{p}$; whatelse it can be ?)
- it doesnot commute with $\hat{x}$,

Dimension of $\hat{k}$: $\left(\frac{1}{L}\right)$

momentum : $\frac{ML}{T}$

we need a conversion factor with dimensionality $\left[\frac{ML^2}{T}\right]$, the reduced Planck const $\hbar$

SI unit $\hbar = 1.055 \times 10^{-34}$ J s $\quad [J] = [E] = \left(\frac{ML^2}{T^2}\right)$

$\Rightarrow \boxed{\hat{p} \equiv \hbar \hat{k}}$ the definition of momentum in QM. (the sign is a convention)

* The definition is beyond the postulates of QM. we are looking for a vector operator, and with the help of CM.
* $\hbar$ has to be ~~measu~~ determined by exp.
* The universality must be examined by exp.

* Anyway, the above definition gives :

$$\lambda = \frac{2\pi\hbar}{p} \quad \text{(de Broglie wavelength)}$$

this can be tested by ~~exp~~ Bragg diffraction. and eventually confirmed

Silicon - n interferometer $\Rightarrow$ phase shift in gravitational field in 1970's

That $\hbar$ is universal was known to Planck (& others) in the 1890's in the expression ~~of~~ for the spectrum of black body radiation.

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Also, Planck realized that $h = \hbar(2\pi)$, c, G could be combined to form a unit for distance, time & mass.

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} = 1.6\times 10^{-33}\ \text{cm}. \qquad t_p = \frac{\ell_p}{c} \simeq 10^{-44}\ \text{sec}.$$

$$m_p \sim 10^{19}\ \text{GeV}.$$

$\Rightarrow$ G.U.T.

$$T(\vec{a}) = \exp\left(\frac{-i}{\hbar}(\vec{a}\cdot\hat{\vec{p}})\right) = \mathbb{1} - \frac{i}{\hbar}\,\vec{a}\cdot\hat{\vec{p}} + \cdots$$

All we have learnt:

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij} \implies \Delta x \Delta p \geq \frac{\hbar}{2}$$

These are the Heisenberg-Born commutation relations.

$$T(\vec{a})|\vec{x}\rangle = \overbrace{\left(\mathbb{1} - \frac{i}{\hbar}\vec{a}\cdot\hat{\vec{p}} + \cdots\right)}^{\text{Taylor expansion of operator}}|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle$$

$$= |\vec{x}\rangle + \vec{a}\cdot\vec{\nabla}|x\rangle + \cdots \quad \text{Taylor expansion of the ket}$$

$$\Rightarrow \boxed{\hat{\vec{p}}|\vec{x}\rangle = i\hbar\vec{\nabla}|\vec{x}\rangle}$$

$\hat{p}$ on $|\psi\rangle$? (HW: prove that $\hat{p}$ is Hermitian)

$$\underline{\langle x|\hat{p}}|\psi\rangle = \underline{-i\hbar\vec{\nabla}\langle x|}\psi\rangle = -i\hbar\vec{\nabla}\psi(x)$$

$\Rightarrow$ We arrive the familiar result that

$\hat{\vec{p}}$ is represented by the operator $(-i\hbar\vec{\nabla})$ in the position representation

$$(\hat{p}\psi)(x) = -i\hbar\vec{\nabla}\psi(x)$$

remember that operator can have many diff representation!!!

usually, in a very confusing way

$$\hat{p}\psi(x) = -i\hbar\vec{\nabla}\psi(x) \quad (\ \times\ ) \quad \text{mess up things!!}$$

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The momentum Representation

$\hat{\vec{p}}$ is a vector of commuting, Hermitian operators

$\rightarrow$ possess a simultaneous eigenbasis,

We write $\quad \hat{\vec{p}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$ → eigenvalues, → eigen ket

In the configuration representation, $\boxed{\psi_p(x) \equiv \langle x|p\rangle}$

$$\langle \vec{x}|\,\hat{\vec{p}}\,|\vec{p}\rangle = -i\hbar\vec{\nabla}\langle\vec{x}|\vec{p}\rangle = -i\hbar\vec{\nabla}\psi_p(x)$$

$$\overset{(-\#)}{=} \vec{p}\langle\vec{x}|\vec{p}\rangle = \vec{p}\,\psi_p(x)$$

$\Rightarrow$ $\psi_p(x)$ is the eigenfunction of the operator $(-i\hbar\vec{\nabla})$ with eigenvalue $\vec{p}$

$$\Rightarrow \boxed{\psi_{\vec{p}}(\vec{x}) = A\,e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}}}$$ is the solution $\quad$ A: a normalization plus a phase.

we pick the normalization $\quad \langle\vec{p}_1|\vec{p}_2\rangle = \delta^3(\vec{p}_1-\vec{p}_2)$

& because: $\displaystyle\int d^3\vec{x}\; e^{i(\vec{p}_2-\vec{p}_1)\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{p}_1-\vec{p}_2)$ $\quad$ from Fourier traf.

or $\displaystyle\int d^3\vec{x}\, e^{i\frac{(\vec{p}_2-\vec{p}_1)\cdot\vec{x}}{\hbar}} = (2\pi\hbar)^3\delta^3(\vec{p}_1-\vec{p}_2)$

$$\Rightarrow \boxed{\langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\,e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}}}$$ a plane wave, with de Broglie wave length

momentum eigenstates are not degenerate. $\hat{\vec{p}}$ (operator) form a complete set of commuting observables.
$\Rightarrow$ momentum representation

$$\boxed{\mathbb{1} = \int d^3\vec{p}\;|\vec{p}\rangle\langle\vec{p}|}$$, the wave function of the state $|\psi\rangle$ in the momentum representation is.

$$\boxed{\phi(\vec{p}) \equiv \langle\vec{p}|\psi\rangle}$$

so. $\quad |\psi\rangle = \int d^3\vec{p}\;|\vec{p}\rangle\langle\vec{p}|\psi\rangle = \int d^3\vec{p}\;|\vec{p}\rangle\,\underbrace{\phi(\vec{p})}_{\text{expansion coefficients in this representation.}}$$

$$\psi(x) \equiv \langle x|\psi\rangle = \int d^3\vec{p}\ \langle x|p\rangle\langle p|\psi\rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi\hbar)^{\frac{3}{2}}}\ e^{i\frac{\vec{p}\cdot\vec{x}}{\hbar}}\ \phi(\vec{p})$$

& its inverse

$$\phi(\vec{p}) = \int \frac{d^3\vec{x}}{(2\pi\hbar)^{\frac{3}{2}}}\ e^{\frac{-i\vec{p}\cdot\vec{x}}{\hbar}}\ \psi(x)$$

(from $\langle p|\psi\rangle = \int d^3\vec{x}\ \langle p|x\rangle\langle x|\psi\rangle$)

$\boxed{\phi(\vec{p}) \ \&\ \psi(x) \text{ are Fourier transforms of one another}}$

In the momentum representation, $\hat{\vec{p}}$ (operator) $\Rightarrow \vec{p}$ (a c-#)

$$\langle\vec{p}|\hat{p}|\psi\rangle = \vec{p}\langle\vec{p}|\psi\rangle = \vec{p}\,\phi(\vec{p})$$

or $(\hat{\vec{p}}\,\phi(\vec{p}) = \vec{p}\,\phi(\vec{p}))$

similarly, $\hat{\vec{x}}$ in the momentum representation

$$\boxed{\langle\vec{p}|\hat{x}|\psi\rangle = i\hbar\frac{\partial\phi(\vec{p})}{\partial\vec{p}}}$$

c.f. $\Leftrightarrow \langle\vec{x}|\hat{\vec{p}}|\psi\rangle = -i\hbar\vec{\nabla}\psi(\vec{x})$

$\because$

$$\langle\vec{p}|\hat{x}|\psi\rangle = \int d^3\vec{x}\ \langle p|x\rangle\langle x|\hat{x}|\psi\rangle = \int\frac{d^3\vec{x}}{(2\pi\hbar)^{\frac{3}{2}}}\ e^{-i\frac{\vec{p}\cdot\vec{x}}{\hbar}}\ x\,\psi(x)$$

$$= i\hbar\frac{\partial}{\partial\vec{p}}\int\frac{d^3\vec{x}}{(2\pi\hbar)^{\frac{3}{2}}}\ e^{\frac{-i\vec{p}\cdot\vec{x}}{\hbar}}\ \psi(x) = i\hbar\frac{\partial}{\partial\vec{p}}\phi(\vec{p})$$

★ Multiparticle W.F.

$$\psi(\vec{x}_1 \cdots \vec{x}_N) \equiv \langle\vec{x}_1, \vec{x}_2 \cdots \vec{x}_N|\psi\rangle$$

$$\phi(\vec{p}_1 \cdots \vec{p}_N) \equiv \langle\vec{p}_1, \vec{p}_2 \cdots \vec{p}_N|\psi\rangle$$