Quantum Mechanics - I

Homepage : http://apctp.org
E-mail : sec@apctp.org

Oct. 28. 2009

Asia Pacific Center for Theoretical Physics

Classical Mechanics (very brief review :)

☆ $\vec{F} = m\vec{a} = m\ddot{x}$

☆ $\boxed{\text{least Action principle \& Lagrangian}}$

An equivalent description of $\vec{F} = m\vec{a}$ is to require that the classical path makes the action extremum;

$$\underline{\underline{S[q_i(t)]}} = \int_{t_i}^{t_f} L(q_i(t), \dot{q}_i(t))\, dt$$

functional : a function of function.
trajectory $q(t)$ is a function of time.

$$\delta S = \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_i}^{t_f}}_{=0,}$$

fixed ~~q_i & q_f~~
$\delta q_i(t_i) = \delta q_i(t_f) = 0$

$\Rightarrow$ Euler - Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$P_i \equiv \dfrac{\partial L}{\partial \dot{q}_i}$ the generalized momentum conjugate to q_i

☆ Lagrange $L = T - U$

say $L = \frac{m}{2}\dot{q}^2 - V(q) \longrightarrow \frac{d}{dt}(m\dot{q}) + \frac{\partial V}{\partial q} = 0$

$\Rightarrow m\ddot{q} = -\frac{\partial V}{\partial q}$ nothing but $F = ma$

☆ So, what's the fuss ?

Symmetry is more apparent, easy to see conservation law, $\Rightarrow$ essential for path integral & QFT !!

Headquarters
Hogil Kim Memorial Building #501, POSTECH, San 31 Hyoja-dong, Namgu
Pohang, Gyeongbuk 790-784, Korea
Tel : +82-54-279-8661~4 Fax : +82-54-279-8679

Branch Office
The Korean Federation of Science and Technology Societies Building 11th floor
635-4 Yoksam-dong, Kangnam-gu, Seoul 135-703, Korea
Tel : +82-2-561-7641~2 Fax : +82-2-561-7140

②

Homepage : http://apctp.org
E-mail : sec@apctp.org

Asia Pacific Center for Theoretical Physics

eg. $m\vec{a} = \frac{Ze\vec{r}}{r^3}$, covariant under rotation.

but $L = \frac{m}{2}\dot{\vec{x}}^2 + \frac{Ze^2}{r}$ is invariant under rotation.

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + \frac{Ze^2}{r}$$

$$\Rightarrow \begin{cases} m\ddot{r} - (mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{Ze^2}{r^2}) = 0 \\ m\frac{d}{dt}(r^2\dot{\theta}) - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0 \\ m\frac{d}{dt}(r^2\sin^2\theta\,\dot{\phi}) = 0 \end{cases} \Rightarrow L_z \text{ is conserved!}$$

① In path integral formulation (later)

$\Rightarrow \sum e^{i\frac{S(\gamma(t))}{\hbar}}$, when $\hbar \to 0$ go back to CM.

② A more profound example of the use of Lagrangian formalism is E&M

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$\rightarrow$ U(1) invariant, all 4 Maxwell equations can be derived from this simple $\mathcal{L}$.

③ ~~Noether's theorem~~ Lagrangian multiplier,
eg. move along a circle

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}r^2\dot{\theta}^2 \qquad x^2 + y^2 - r^2 = 0$$

$\rightarrow$ E.O.M gives $\ddot{\theta} = 0$, $\neq \{\ddot{x} = 0 \text{ \& } \ddot{y} = 0\}$

$$L' = L + \lambda f(x, y)$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \lambda(x^2 + y^2 - r^2)$$

$$\Rightarrow \begin{cases} m\ddot{x} + 2\lambda x = 0 \rightarrow \lambda = -\frac{m\ddot{x}}{2x} = -\frac{m\ddot{x}}{2\sqrt{r^2 - y^2}} \\ m\ddot{y} + 2\lambda y = 0 \\ x^2 + y^2 - r^2 = 0 \end{cases}$$

$$\frac{\ddot{y}}{y} - \frac{\ddot{x}}{x} = 0$$

or $\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0 \Rightarrow$ conservation of angular momentum

Headquarters
Hogil Kim Memorial Building #501, POSTECH, San 31 Hyoja-dong, Namgu
Pohang, Gyeongbuk 790-784, Korea
Tel : +82-54-279-8661~4 Fax : +82-54-279-8679

Branch Office
The Korean Federation of Science and Technology Societies Building 11th floor
635-4 Yoksam-dong, Kangnam-gu, Seoul 135-703, Korea
Tel : +82-2-561-7641~2 Fax : +82-2-561-7140

Homepage : http://apctp.org
E-mail : sec@apctp.org

Asia Pacific Center for Theoretical Physics

☆ Noether's theorem

If S is invariant under the transformation $q_i \to q_i + \epsilon f_i(q, \dot{q})$

There is a conserved charge Q.

$$\delta S = \int dt \left(\delta q_i \frac{\partial L}{\partial q_i} + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) = \int dt \left\{ \delta q_i \left[\underbrace{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right)}_{=0 \text{ by E.O.M}} \right] + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}} \delta q_i \right] \right\}$$

(hold for any δq)

$$\Rightarrow \delta S = \int dt \, \epsilon \dot{J}$$ corresponds to (some symmetry)

$$\Rightarrow \quad Q \equiv f_i(q) \frac{\partial L}{\partial \dot{q}_i} - J$$ is a const of motion.

$\boxed{\text{☆ Conservation of energy}}$

$t \to t + \epsilon \quad , \quad \delta q = \epsilon \dot{q}$

$$\Rightarrow \delta S = \int dt \left(\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} \right) = \int dt \, \epsilon \left(\dot{q} \frac{\partial L}{\partial q} + \ddot{q} \frac{\partial L}{\partial \dot{q}} \right) = \int dt \, \epsilon \frac{dL}{dt}$$

$\Rightarrow$ time translation invariant $\Rightarrow J = L$

$$\Rightarrow H \equiv \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = p_i \dot{q}_i - L$$ is a const

Energy / Hamiltonian

$\boxed{\text{☆ Conservation of momentum}}$

$\delta q = \epsilon$, & the symmetry requires that $L = L(\dot{q})$, q-independent

$$\Rightarrow \delta S = \int dt \left(\epsilon \cancel{\frac{\partial L}{\partial q}} + \underbrace{\delta \dot{q}}_{\dot{\epsilon} = 0} \frac{\partial L}{\partial \dot{q}} \right) = 0 \quad \Rightarrow J = 0$$

$$\Rightarrow \quad Q = \frac{\partial L}{\partial \dot{q}_i} = p_i$$ momentum conservation

Headquarters
Hogil Kim Memorial Building #501, POSTECH, San 31 Hyoja-dong, Namgu
Pohang, Gyeongbuk 790-784, Korea
Tel : +82-54-279-8661~4 Fax : +82-54-279-8679

Branch Office
The Korean Federation of Science and Technology Societies Building 11th floor
635-4 Yoksam-dong, Kangnam-gu, Seoul 135-703, Korea
Tel : +82-2-561-7641~2 Fax : +82-2-561-7140

$$dH = \sum(p_i d\dot q_i + \dot q_i dp_i) - \underbrace{dL}_{\sum_i\left(\frac{\partial L}{\partial q_i}dq_i + p_i d\dot q_i\right)} = \sum(\dot q_i dp_i - \dot p_i dq_i)$$

Homepage : http://apctp.org
E-mail : sec@apctp.org

Asia Pacific Center for Theoretical Physics

$\boxed{\text{Hamiltonian}}$ $\quad L(q, \dot q) \rightarrow H(q, p)$

canonical $\quad p_i = \left.\frac{\partial L}{\partial \dot q_i}\right|_{q_i}$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot q}\frac{d\dot q}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\dot q = \frac{d}{dt}(p\dot q) - \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\dot q}$$

Legendre transformation

$$\Rightarrow \frac{d}{dt}(p\dot q - L) = 0$$

just the total energy. but more than that

internal energy $U(S, V)$

$$T = \left.\frac{\partial U}{\partial S}\right|_V \longrightarrow F(T, V) \quad \text{free energy}$$

$$dU = \underbrace{\frac{\partial U}{\partial S}}_{T} dS + \frac{\partial U}{\partial V} dV = d(TS) - S\,dT + \frac{\partial U}{\partial V} dV$$

$$\Rightarrow d(\underbrace{U - TS}_{F}) = -S\,dT + \frac{\partial U}{\partial V} dV \Rightarrow S = -\left.\frac{\partial F}{\partial T}\right|_V$$

$$H(p, q) = \sum p\dot q - L(\dot q, q) \Rightarrow \text{function of } p \,\&\, q, \ \underline{\text{No}} \text{ time derivatives}$$

$$\Rightarrow \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad \frac{d}{dt}q_i = \left.\frac{\partial H}{\partial p_i}\right|_{q_i}$$

: Hamilton equation of motion. (1st order D.E.)

$\Rightarrow$ Very useful to study $\begin{cases}\text{chaos, statistical mechanics} \\ \text{nonlinear dynamics, beam dynamics}\end{cases}$

$\boxed{\text{Poisson Bracket}}$

$$\{A, B\}_p \equiv \sum\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right), \quad A, B: \text{functions of } p, q$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\dot q_i + \frac{\partial A}{\partial p_i}\dot p_i + \frac{\partial A}{\partial t} \quad = 0 \text{ if no explicit time dependence.}$$

$$= \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial A}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = \{A, H\}_p$$

Just like the momentum is the generator for traslation
Hamiltonian is the generator for time evolution

1925, Dirac

$$\text{CM} \quad \{\ ,\ \}_p \ \underset{\text{only 1-way}}{\longleftarrow} \ \frac{[\ ,\ ]}{i\hbar} \quad \text{QM}$$

Headquarters
Hogil Kim Memorial Building #501, POSTECH, San 31 Hyoja-dong, Namgu
Pohang, Gyeongbuk 790-784, Korea
Tel : +82-54-279-8661~4 Fax : +82-54-279-8679

Branch Office
The Korean Federation of Science and Technology Societies Building 11th floor
635-4 Yoksam-dong, Kangnam-gu, Seoul 135-703, Korea
Tel : +82-2-561-7641~2 Fax : +82-2-561-7140

Homepage : http://apctp.org
E-mail : sec@apctp.org

Asia Pacific Center for Theoretical Physics

⊗ Hamilton – Jacobi Equation

from an example, free particle in 1-D.

$$L = \frac{m}{2}\dot{x}^2, \qquad \begin{cases} t_f : x_f \\ t_i : x_i \end{cases} \Rightarrow \quad x(t) = x_i + \frac{x_f - x_i}{t_f - t_i}(t - t_i)$$

or $$S(x_f, t_f; x_i, t_i) = \frac{m}{2}\frac{(x_f - x_i)^2}{t_f - t_i}$$

It's interesting to note that

$$\frac{\partial S}{\partial x_f} = m\frac{x_f - x_i}{t_f - t_i} = p, \quad \frac{\partial S}{\partial t_f} = -\frac{m}{2}\frac{(x_f - x_i)^2}{(t_f - t_i)^2} = -E$$

so $$E = H(p, q) = \frac{p^2}{2m}$$

$$\frac{\partial S}{\partial t_f} + \frac{1}{2m}\left(\frac{\partial S}{\partial x_f}\right)^2 = 0$$

In general, regard the action as a function of the final position q_f and time t_f, keeping the initial data fixed,

$$\Rightarrow \quad \frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial t} = -H$$ (remind you the QM correspondence, right?)

This is the so called Hamilton – Jacobi eq.

$$\boxed{\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0}$$

(you can test its power by solving the kelper problem.)

See: { Shankar chap. 2
Landau Mechanics, chap. 7

Homepage : http://apctp.org
E-mail : sec@apctp.org

Derivation

$\delta q_i(t_i) = 0$, $\delta q_i(t_f) = \delta q_i$ and the path changes accordingly

$$\delta S = \int_{t_i}^{t_f} \Big(\underbrace{\frac{\partial L}{\partial q_i}}_{\Downarrow} \delta q_i + \frac{\partial L}{\partial \dot q_i} \delta \dot q_i \Big) dt$$

$$= \int_{t_i}^{t_f} \Big(\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q_i}\Big) \delta q_i + \frac{\partial L}{\partial \dot q_i} \delta \dot q_i \Big) dt \qquad \text{by E.O.M.}$$

$$= \Big(\frac{\partial L}{\partial \dot q_i} \delta q_i \Big)\Big|_{t_i}^{t_f} = \frac{\partial L}{\partial \dot q_i}(t_f)\, \delta q_i$$

$$\Rightarrow \boxed{\frac{\partial S}{\partial q_i} = p_i(t_f)}$$

Next, we fix $q_i(t_f)$ and vary $t_f \to t_f + \delta t$

so. $q_i'(t_f + \delta t) = q_i(t_f)$

or $q_i(t_f) \to q_i(t_f) - \dot q_i \delta t \quad \Rightarrow \delta q_i(t_f) = -\dot q_i \delta t$

$$\Rightarrow \delta S = L(t_f)\delta t + \frac{\partial L}{\partial \dot q_i}(t_f) \delta q_i = L(t_f)\delta t - \frac{\partial L}{\partial \dot q_i}(t_f)\, \dot q_i\, \delta t$$

$$= -H \delta t \qquad \Rightarrow \boxed{\frac{\partial S}{\partial t} = -H}$$

※ Why doing this?

Useful in dealing some complicated problem.

like Kepler motion, hydrogen atom

The Hamilton – Jacobi eq. makes the mechanics problem <u>mechanical</u>.

If variables can be separated. $\Rightarrow$ problem reduces to simple integrals.

Headquarters
Hogil Kim Memorial Building #501, POSTECH, San 31 Hyoja-dong, Namgu
Pohang, Gyeongbuk 790-784, Korea
Tel : +82-54-279-8661~4 Fax : +82-54-279-8679

Branch Office
The Korean Federation of Science and Technology Societies Building 11th floor
635-4 Yoksam-dong, Kangnam-gu, Seoul 135-703, Korea
Tel : +82-2-561-7641~2 Fax : +82-2-561-7140