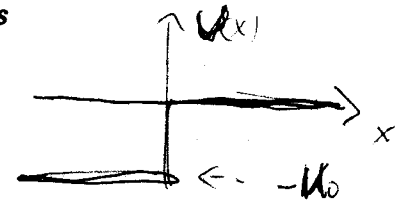
Nov. 11. 2009



Some simple 1D QM problem

* Potential step

$$-\psi'' + U(x)\psi = \varepsilon\psi \quad , \left(\varepsilon = \frac{2mE}{\hbar^2}, \quad U(x) = \frac{2mV(x)}{\hbar^2} \right)$$

① $\varepsilon > 0$, $\quad \psi'' + (\varepsilon - U)\psi = 0$

$x < 0$, $\quad \psi_-'' + k_1^2 \psi_- = 0, \quad k_1^2 \equiv \varepsilon + U_0$

$x > 0$ $\quad \psi_+'' + k_2^2 \psi_+ = 0, \quad k_2^2 \equiv \varepsilon$

$$\Rightarrow \psi_- = A e^{ik_1 x} + B e^{-ik_1 x}$$
$$\psi_+ = C e^{ik_2 x} + D e^{-ik_2 x}$$

Imagine a wave $e^{ik_1 x}$ coming from the left, $\Rightarrow A = 1$, $B \neq 0$ (reflected wave)

$C \neq 0$ (transmitted wave), $D = 0$ (no wave coming in from the right)

$$\psi_- = e^{ik_1 x} + B e^{-ik_1 x}$$
$$\psi_+ = C e^{ik_2 x}$$

WF and ψ' continuous at $x = 0$ $\Rightarrow \begin{cases} 1 + B = C \\ ik_1(1 - B) = ik_2 C \end{cases}$

$$\Rightarrow \psi_- = e^{ik_1 x} + \frac{k_1 - k_2}{k_1 + k_2} e^{-ik_1 x}, \quad \psi_+ = \frac{2k_1}{k_1 + k_2} e^{ik_2 x}$$

check it's probability current for the incident wave

$$j_{inc} = \frac{\hbar k_1}{m} = \frac{p_1}{m} = v_1, \quad j_{refl} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \frac{\hbar k_1}{m},$$

The reflection coefficient $R = \left| \frac{j_{refl}}{j_{inc}} \right| = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$

& $j_{transmitted} = \frac{4k_1^2}{(k_1 + k_2)^2} \frac{\hbar k_2}{m}$, $\quad T = \left| \frac{j_{trans}}{j_{inc}} \right| = \frac{4k_1 k_2}{(k_1 + k_2)^2}$

$\Rightarrow R + T = 1$ / $\varepsilon \to 0$, $k_2^2 \to 0$, $R \to 1$, $T \to 0$ /

$\varepsilon \to \infty$ $\quad k_2^2 \to k_1^2 \Rightarrow R \to 0, \ T \to 1$

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Now consider the initial wave coming in from the right

$$\Rightarrow \begin{cases} \psi_+ = C' e^{ik_2 x} + e^{-ik_2 x} \\ \psi_- = B' e^{-ik_1 x} \end{cases} \Rightarrow B' = \frac{2k_2}{k_1+k_2}, \quad C' = \frac{k_2-k_1}{k_1+k_2}$$

$$\Rightarrow T = \frac{4k_2k_1}{(k_1+k_2)^2}$$, same as before. and EM wave

vacuum | dielectric

(2) $\boxed{-V_0 < \varepsilon < 0}$, we write $k_2^2 = -\varepsilon > 0$

$x<0$, $\quad \psi_-'' + k_1^2 \psi_- = 0$

$x>0$ $\quad \psi_+'' - k_2^2 \psi_+ = 0$

An incident wave (e^{ik_1x}) from the left.

$$\psi_- = e^{ik_1x} + Be^{-ik_1x}, \quad \psi_+ = Ce^{-k_2x}$$

$$\Rightarrow B = \frac{k_1 - ik_2}{k_1 + ik_2}, \quad C = \frac{2k_1}{k_1+ik_2}$$

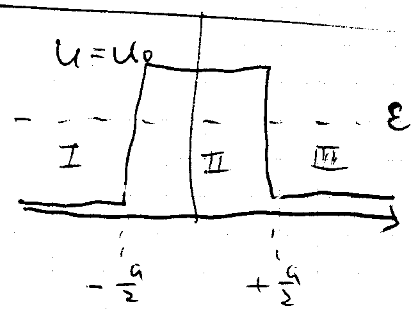
$(\vec{j} = 0)$

$$\Rightarrow \begin{cases} \psi_- = \frac{2k_1}{k_1+ik_2}\cos k_1 x - \frac{2k_2}{k_1+ik_2}\sin k_1 x \\ \psi_+ = \frac{2k_1}{k_1+ik_2} e^{-k_2x} \end{cases}$$

$\Rightarrow$ standing wave in the classical allowed region

$\Rightarrow$ exponentially decaying profile in the classically forbidden region.

1-D rectangular Barrier



for I, III $\quad u'' + k_1^2 u = 0 \qquad k_1^2 = \varepsilon > 0$

II $\quad u'' - k_2^2 u = 0 \qquad k_2^2 = U_0 - \varepsilon > 0$

For a wave incident from the left.

I : $u = e^{ik_1x} + Be^{-ik_1x}$

II : $u = Ce^{k_2x} + De^{-k_2x}$

III : $u = Se^{ik_1x}$

match at $x = \pm\frac{a}{2}$

$$\Rightarrow S = \frac{4ik_1k_2\, e^{-ik_1a}}{e^{-k_2a}(k_2+ik_1)^2 - e^{k_2a}(k_2-ik_1)^2}$$

$a \to 0$, $S \to 1$

For $(k_2 a \gg 1)$, the transmission coefficient $T = |S|^2 \simeq \frac{16k_1^2k_2^2}{(k_1^2+k_2^2)^2} e^{-2k_2a}$

$\Rightarrow$ barrier penetration

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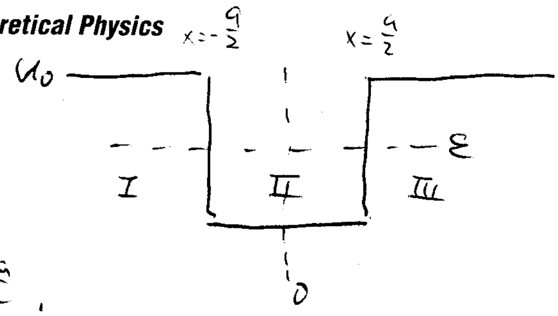


☆ 1-D rectangular well

Since the $V(x)$ is symmetric, we only need to consider the solutions in regions II, III and B.C at $x=\frac{a}{2}$.

II: $\quad u'' + k_1^2 u = 0 \qquad k_1^2 = \varepsilon$

III: $\quad u'' - k_2^2 u = 0 \qquad k_2^2 = U_0 - \varepsilon$

$u_{\text{II, even}} = A\cos k_1 x$, $\quad U_{\text{II, odd}} = B\sin k_1 x$

(when $U_0 \to \infty$. $\quad u = 0$ for $|x| \geq \frac{a}{2}$

$\Rightarrow \begin{cases} \cos\frac{k_1 a}{2} = 0, & \varepsilon_{even} = \frac{\pi^2}{a^2}, \frac{9\pi^2}{a^2}, \frac{25\pi^2}{a^2} \cdots \\ \sin\frac{k_1 a}{2} = 0, & \varepsilon_{odd} = \frac{4\pi^2}{a^2}, \frac{16\pi^2}{a^2}, \frac{36\pi^2}{a^2} \cdots \end{cases}$)

consider the Lowest energy solution for the finite well.

$\Rightarrow \quad |x| < \frac{a}{2}, \quad u = A\cos k_1 x$

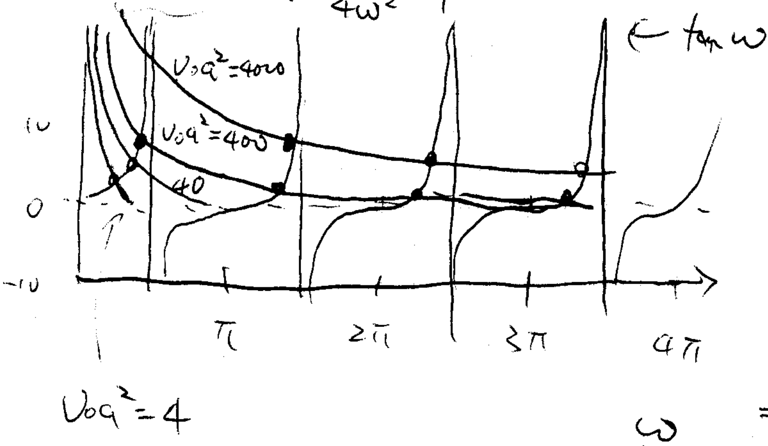
$\quad x > \frac{a}{2} \quad u = B\exp(-k_2 x)$

$\Rightarrow$ B.C $\quad A\cos\frac{k_1 a}{2} = B\exp(-\frac{k_2 a}{2})$

$\quad -k_1 A \sin\frac{k_1 a}{2} = -k_2 B\exp(-\frac{k_2 a}{2})$

or $\tan\frac{k_1 a}{2} = \frac{k_2}{k_1}$, $\rightarrow \tan\sqrt{\frac{\varepsilon a^2}{4}} = \sqrt{\frac{U_0}{\varepsilon} - 1}$, $\quad \omega^2 \equiv \frac{\varepsilon a^2}{4}$

$\Rightarrow \tan\omega = \sqrt{\frac{U_0 a^2}{4\omega^2} - 1}$



$\Rightarrow$ There always exist a solution for any $U_0 a^2$ (not for 3-D!)

$\Rightarrow$ No matter how shallow the well there is always at least one bound state (even)!!

$\Rightarrow$ However for odd ones

$-\cot\omega = \sqrt{\frac{U_0 a^2}{4\omega^2} - 1}$

unless $U_0 \geq \left(\frac{\pi}{a}\right)^2$, $\Rightarrow$ no solution!!

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Now, consider the cases of $\varepsilon > 0$.

$k_1^2 \equiv \varepsilon > 0, \quad k_2^2 \equiv \varepsilon + U_0 > 0$

A wave e^{ikx} incident from the left.

$$\begin{cases} x < -\frac{a}{2} & u = e^{ik_1 x} + B e^{-ik_1 x} \\ |x| < \frac{a}{2} & u = C e^{ik_2 x} + D e^{-ik_2 x} \\ x > \frac{a}{2} & u = S e^{ik_1 x} \end{cases}$$

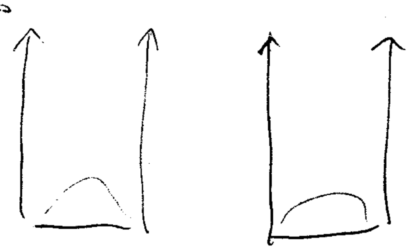
match the WFs at $x = \pm\frac{a}{2}$, $\Rightarrow S = \dfrac{2e^{-ik_1 a}}{2\cos k_2 a - \frac{i(k_1^2+k_2^2)}{k_1 k_2}\sin k_2 a}$

and the transmission coefficient is:

$$T = |S|^2 = \frac{1}{\cos^2(k_2 a) + \frac{(k_1^2+k_2^2)^2}{4k_1^2 k_2^2}\sin^2(k_2 a)}$$

Note: $T = 1$, when $\underline{k_2 a = n\pi}$. this is the resonance scattering!

Double Wells. (totally disconnected)



$\psi_L \quad \psi_R$

* Since the well is infinitely deep one can have 2-fold degeneracy.

& $\psi_+ = \frac{1}{\sqrt{2}}(\psi_L + \psi_R)$
$\psi_- = \frac{1}{\sqrt{2}}(\psi_L - \psi_R)$ } with same eigenenergy

* Now let the central barrier finite

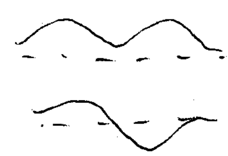
$\Rightarrow E_- > E_+$

still we can define

$\psi_R' \equiv \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$

& $\psi_L' \equiv \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$

$E_+ : \psi_+$
$E_- : \psi_-$



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Then

$$\psi'_{R/L}(x,t) = \frac{1}{\sqrt{2}}\left[\psi_+(x)\, e^{\frac{-iE_+t}{\hbar}} \mp \psi_-(x)\, e^{\frac{-E_-t}{\hbar}}\right]$$

$\Rightarrow$ Neither ψ'_R nor ψ'_L are stationary states.

~~If at $t=0$, we have~~

~~$\psi'_R(x,0) = \frac{1}{\sqrt{2}}\left[\psi_+(x) - \psi_-(x)\right]$: localized a particle in the right well,~~

at later time t

$$\psi'_R(x,t) = \frac{1}{\sqrt{2}}\left[\psi_+(x)\, e^{\frac{-iE_+t}{\hbar}} - \psi_-(x)\, e^{\frac{-iE_-t}{\hbar}}\right]$$

$$= \frac{1}{\sqrt{2}}\, e^{\frac{-iE_+t}{\hbar}}\left[\psi_+(x) - \psi_-(x)\, e^{\frac{-i(E_- - E_+)t}{\hbar}}\right]$$

when $\frac{(E_- - E_+)t}{\hbar} = \pi$ $\rightarrow$ the particle tunnel over to the left well.

" $= 2\pi$ $\rightarrow$ right

" $= 3\pi$ $\rightarrow$ left

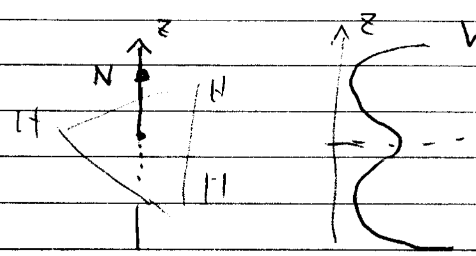
oscillations in time,

$\Rightarrow$ quantum beats;

occur when there is coherent superposition of more than one states with diff. energies.

e.g.

☆ Ammonia molecule

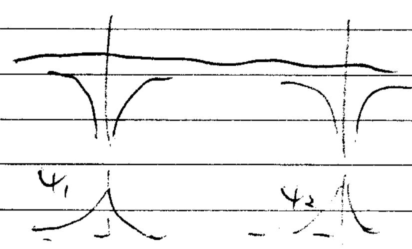


1954, MASER (Micro wave Amplification by Stimulated Emission of Radiation)

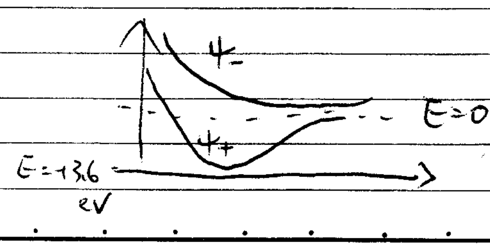
(Nobel 1964) Townes & coworkers

24 GHz

☆ ~~H_2^+ molecule ($p+p+e^-$)~~



$$\psi_\pm = \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2)$$



☆ neutrino oscillation

分類:
編號:
總號:

$$|\nu_1\rangle = \cos\theta\,|\nu_e\rangle + \sin\theta\,|\nu_\mu\rangle$$

$$|\nu_2\rangle = -\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle$$

Say at $t=0$. an electro ν was created

$$|\psi(0)\rangle = |\nu_e\rangle = \cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle$$

$E = \sqrt{p^2c^2+m^2c^4}$, in natural unit $\hbar = c = 1$

$E = \sqrt{p^2+m^2}$

$$E_1 \simeq p + \frac{m_1^2}{2p}, \qquad E_2 \simeq p + \frac{m_2^2}{2p}$$

$$\Rightarrow |\psi(t)\rangle = \cos\theta\,|\nu_1\rangle e^{-iE_1 t} - \sin\theta\,|\nu_2\rangle e^{-iE_2 t}$$

$$\simeq \cos\theta\,|\nu_1\rangle \exp\left(-ipt - \frac{i m_1^2 t}{2p}\right) - \sin\theta\,|\nu_2\rangle \exp\left(-ipt - i\frac{m_2^2 t}{2p}\right)$$

$$= e^{-ipt}\, e^{-\frac{i m_1^2 t}{2p}} \left[\cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle\, e^{-i\frac{(m_2^2-m_1^2)t}{2p}}\right]$$

$$= e^{-ipt}\, e^{-\frac{i m_1^2 t}{2p}} \Big(c^2\theta\,|\nu_e\rangle + s\theta c\theta\,|\nu_\mu\rangle$$

$$+ s^2\theta\,|\nu_e\rangle e^{-i\frac{\Delta m^2 t}{2p}} - s\theta c\theta\,|\nu_\mu\rangle e^{-i\frac{\Delta m^2 t}{2p}} \Big]$$

$$\Rightarrow P(\nu_e \to \nu_\mu)(t) = \left|\langle \nu_\mu | \psi(t)\rangle\right|^2 = s^2\theta\, c^2\theta \left| 1 - e^{-i\frac{\Delta m^2 t}{2p}} \right|^2$$

$$= \left(\frac{1}{4} s^2 2\theta\right) 4 \sin^2 \frac{\Delta m^2 t}{4p} = \sin^2 2\theta\, \sin^2\left(\frac{\Delta m^2 t}{4p}\right)$$

$ct = L$

$$\boxed{P(\nu_e \to \nu_\mu) = \sin^2 2\theta\, \sin^2\left(\frac{\Delta m^2}{4E} L\right)}$$