Quantum Mechanics - I

1D Simple Harmonic Oscillator

☆ $\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$, with $[q, p] = i\hbar$

☆ 2 ways of solving this problem: ① Schrödinger's diff equation
② algebraically. ←
(operator, Dirac)
important in QFT.

$$P \equiv \frac{1}{\sqrt{m\omega\hbar}} p, \quad H \equiv \hbar\omega \hat{H}$$

$$Q \equiv \sqrt{\frac{m\omega}{\hbar}} q$$

$$\Rightarrow \hat{H} = \frac{\hbar\omega}{2}(P^2 + Q^2), \Rightarrow \boxed{H = \frac{1}{2}(P^2+Q^2), \quad [Q, P] = i}$$

☆ Define operators:

$$a = \frac{1}{\sqrt{2}}(Q + iP), \quad a^+ = \frac{1}{\sqrt{2}}(Q - iP)$$

then $[a, a^+] = \frac{1}{2}[Q+iP, Q-iP] = \frac{i}{2}(-[Q,P] + [P,Q]) = 1$

also, $Q = \frac{1}{\sqrt{2}}(a + a^+)$, $P = \frac{-i}{\sqrt{2}}(a - a^+)$

$$\Rightarrow H = \frac{1}{2}\left(\frac{1}{2}(a^2 + (a^+)^2 + aa^+ + a^+a) - \frac{1}{2}(a^2 + (a^+)^2 - aa^+ - a^+a)\right)$$
$$= \frac{1}{2}(aa^+ + a^+a) = (a^+a + \frac{1}{2})$$

$$\Rightarrow \boxed{[a, a^+] = 1, \quad H = a^+a + \frac{1}{2}}$$

☆ consider the operator $N \equiv a^+a$, $\Rightarrow \boxed{H = N + \frac{1}{2}}$

$[N, a] = (a^+aa - aa^+a) = -a$

$[N, a^+] = a^+aa^+ - a^+a^+a = +a^+$ $\Rightarrow \boxed{[a, N] = a, \quad [a^+, N] = -a^+}$

☆ let $|x\rangle$ be a non-zero eigenvector of N with eigenvalue x

$$N|x\rangle = x|x\rangle$$

define $|y\rangle = a|x\rangle$

then $\langle x|N|x\rangle = \langle x|a^{+}a|x\rangle = \langle y|y\rangle = x\langle x|x\rangle$

Thus $\boxed{x > 0}$, unless $|y\rangle = a|x\rangle = 0$, $(x = 0)$

on the other hand,

$N a|x\rangle = (aN - a)|x\rangle = x a|x\rangle - a|x\rangle = (x-1)(a|x\rangle)$

$\Rightarrow (a|x\rangle)$ is also an eigenstate of N with eigenvalue $(x-1)$ (annihilation operator)

moreover,

$N a^{+}|x\rangle = (a^{+}N + a^{+})|x\rangle = (x+1)(a^{+}|x\rangle)$

$\Rightarrow a^{+}|x\rangle$ an eigenstate of N with eigenvalue $(x+1)$ (creation operator

One can repeatedly apply (a), and get

Eigenvector	Eigenvalue
$\|x\rangle$	x
$a\|x\rangle$	$(x-1)$
$a^2\|x\rangle$	$(x-2)$
$\vdots$	$\vdots$
$a^p\|x\rangle$	$(x-p)$

☆ The series must terminate otherwise we will go to the hell ~

It happens only for some eigenvector $|x_0\rangle$ such that $a|x_0\rangle = 0$.

Then $a^n|x_0\rangle$ is still 0 !!

$\Rightarrow$ $\boxed{x \text{ must be nonnegative integer}}$

from now on, we write it as $|n\rangle$

$$\boxed{N|n\rangle = n|n\rangle}$$

and we immediately obtain the eigenenergy of S.H.O.

$$\Rightarrow \boxed{E_n = (n+\tfrac{1}{2})\hbar\omega}$$

And we know that each eigenstate is nondegenerate !

Normalization of eigenvectors.

Let $|0\rangle$ be the eigenvector with eigenvalue $n=0$, also $\langle 0|0\rangle = 1$.

define $|1\rangle = a^{+}|0\rangle$, Then

$$\langle 1|1\rangle = \langle 0|a a^{+}|0\rangle = \langle 0|(1+a^{+}\overbrace{a)|0\rangle}^{=0} = \langle 0|0\rangle = 1$$

consider $|2\rangle \equiv \frac{1}{\sqrt{2}} a^{+}|1\rangle$

$$\langle 2|2\rangle = \frac{1}{2}\langle 1|a a^{+}|1\rangle = \frac{1}{2}\langle 1|(1+a^{+}a)|1\rangle = \frac{1}{2}\{\langle 1|1\rangle + \langle 0|0\rangle\} = 1$$

Next. $|3\rangle \equiv \frac{1}{\sqrt{3}} a^{+}|2\rangle$, also $\langle 3|3\rangle = 1$ . . .

$$|n\rangle = \frac{1}{\sqrt{n}} a^{+}|n-1\rangle = \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n-1}}(a^{+})^{2}|n-2\rangle = \cdots\cdots = \frac{1}{\sqrt{n!}}(a^{+})^{n}|0\rangle$$

Similarly, we have $\boxed{\frac{a^{n}}{\sqrt{n!}}|n\rangle = |0\rangle, \quad \frac{1}{\sqrt{n!}}(a^{+})^{n}|0\rangle = |n\rangle}$

or

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

Thus

$$\langle n'|N|n\rangle = n\,\delta_{n'n}$$
$$\langle n'|a|n\rangle = \sqrt{n}\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n',n-1}$$
$$\langle n'|a^{+}|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

$\langle n'|x|n\rangle$

$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta\right.$

$\langle n'|p|n\rangle$

$= i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n'}\right.$

$\Rightarrow \langle x\rangle = \langle p\rangle = 0$ for any $|n\rangle$

Wave function of the S.H.O.

Let's find the wave function (coordinate rep) of $|n\rangle$.

We have $a|0\rangle = 0$,

$$\Rightarrow 0 = \underbrace{\int dQ'\,|Q'\rangle\langle Q'|}_{=\mathbb{1}}\, a|0\rangle = \frac{1}{\sqrt{2}}\int dQ'\,|Q'\rangle\langle Q'|(Q+iP)|0\rangle$$
$$= \frac{1}{\sqrt{2}}\int dQ'\,|Q'\rangle\left(Q' + \frac{\partial}{\partial Q'}\right)\psi_0(Q')$$

$a = \frac{1}{\sqrt{2}}(Q+iP)$ Times $\langle Q|$ from the left. $(\langle Q|Q'\rangle = \delta(Q-Q'))$

$$\Rightarrow \left(Q + \frac{\partial}{\partial Q}\right)\psi_0(Q) = 0 \quad \Rightarrow \boxed{\psi_0(Q) = \frac{1}{(\pi)^{\frac{1}{4}}}\exp\left(-\frac{Q^2}{2}\right)}$$

This is the ground state wave function for S.H.O.

For $n > 0$,

$$\psi_n(Q) = \langle Q | n \rangle = \langle Q | \frac{(a^\dagger)^n}{\sqrt{n!}} | 0 \rangle$$

$$= \int dQ' \langle Q | \frac{(a^\dagger)^n}{\sqrt{n!}} | Q' \rangle \langle Q' | 0 \rangle$$

$$\boxed{\psi_n(Q) = \frac{1}{(\sqrt{2})^n \sqrt{n!}} \left(Q - \frac{\partial}{\partial Q} \right)^n \psi_0(Q)}$$

You don't need to remember what $H_n(x)$ is. $\quad H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$

☆ SHO and the Heisenberg eq.

In the Heisenberg picture, we have $\hat{O}_H = U^\dagger \hat{O}_S U$, $U = \exp(-i\frac{Ht}{\hbar})$

and $$\frac{d\hat{O}_H}{dt} = \frac{1}{i\hbar} [\hat{O}_H, \hat{H}]$$

Now, let's apply this to $a_H \equiv U^\dagger a U$. We have

$$\frac{d a_H}{dt} = \frac{1}{i\hbar}[a_H, H] = \frac{1}{i\hbar}[a_H, \hbar\omega(N+\tfrac{1}{2})] = \frac{\hbar\omega}{i\hbar}[a_H, N]$$

$$= -i\omega(U^\dagger a U N - N U^\dagger a U) \qquad [N, H] = 0$$

$$= -i\omega(U^\dagger a N U - U^\dagger N a U) = -i\omega U^\dagger [a, N] U = -i\omega a_H$$

or $$\boxed{\frac{d a_H}{dt} = -i\omega a_H}$$

similarly $$\boxed{\frac{d a_H^\dagger}{dt} = +i\omega a_H^\dagger}$$

$$\Rightarrow \quad a_H(t) = a_H(0)\, e^{-i\omega t}, \qquad a_H^\dagger(t) = a_H^\dagger(0)\, e^{+i\omega t}$$

However, $Q_H = \frac{1}{\sqrt{2}}(a_H + a_H^\dagger)$, $P_H = \frac{i}{\sqrt{2}}(a_H^\dagger - a_H)$

$$\Rightarrow \begin{cases} Q_H(t) = \cos\omega t\, Q_H(0) + \sin\omega t\, P_H(0) \\ P_H(t) = \cos\omega t\, P_H(0) - \sin\omega t\, Q_H(0) \end{cases}$$

or go back to p & q

$$\rightsquigarrow \begin{cases} q_H(t) = q_H(0)\cos\omega t + \frac{1}{m\omega} P_H(0) \sin\omega t \\ P_H(t) = P_H(0)\cos\omega t - m\omega\, q_H(0) \sin\omega t \end{cases}$$