Quantum Mechanics — I

☆ **Path-Integral** (a prelude) see Rev Mod. Phys. 20, 367

In short, $\overbrace{\langle x_f, t_f | x_i, t_i \rangle}^{\text{amplitude}} = \int \mathcal{D}x(t)\; e^{\,i\frac{S[x(t)]}{\hbar}}$

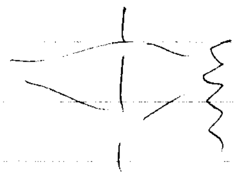
where $S[x(t)] = \int_{x_i}^{x_f} dt\; \mathcal{L}(x(t))$ is the classical action for each path $x(t)$. Hence the name path integral.

☆ $|x\rangle$ form a complete set of basis, knowing this amplitude for all x is enough information to tell you ~~say~~ everything about the system!

① $\hbar \to 0$ limit, is classical Mechanics, $\Rightarrow$ Euler-Lagrange Eq.

② We don't know what ~~path~~ path the particle has chosen

$\rightarrow$ infinite-slit experiment



③ provide intuition on what quantum fluctuation does.
Around the classical trajectory, a quantum particle explores the vicinity $\sim \hbar$
tunneling, fall down from the top of hill.

④ Integral expression for a quantity $\rightarrow$ easier to have an approximation method to work it out $\Longleftrightarrow$ D.E. eg. perturbative expansion
steepest descent method.

⑤ Can also be used to calculate partition functions in statistical mechanics

⑥ It's easier to see the connection between the symmetry, conservation law and unitary transformation.

⑦ Lorentz invariant $\Longleftrightarrow$ useful in Quantum field Theory
In fact, it's the only way to quantize nonabelian YM theory.

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Downsides :

- The ~~actural~~ actual calculation of P^2 is technical and awkward.
- Mathematically well-defined? (yes, it is —)
- Hard to read off energy eigenvalues $\longleftrightarrow$ Schrödinger Eq. (bound state)

The propagator for the Schrödinger eq.

$$H(t)\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right)\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

The Hamiltonian is associated with a time-evolution operator $U(t,t_0)$ with B.C. $U(t_0,t_0) = \mathbb{1}$.

The propagator is a function of two space-time points.

$$K(x,t;x_0,t_0) \equiv \langle x \,|\, U(t,t_0) \,|\, x_0\rangle$$

is just the x-space matrix element of $U(t,t_0)$ between x_0 and x

$\sim$ the amplitude to find the particle at (x,t) given that it was at (x_0,t_0)

* It's closely related to time-dependent Green's functions. (later in scattering theory)

$$K(x\,t_0;x_0t_0) = \langle x|\,U(t_0,t_0)=\mathbb{1}\;|x_0\rangle = \delta(x-x_0)$$

Also,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}K(x\,t;x_0t_0) &= \langle x|\,i\hbar\frac{\partial}{\partial t}U(t,t_0)\,|x_0\rangle \\ &= \langle x|\,\hat{H}(t)\,U(t,t_0)\,|x_0\rangle \\ &= H(t)\,\langle x\,|\,U(t,t_0)\,|x_0\rangle \\ &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right)K(x,t;x_0t_0) \end{aligned}$$

(x_0,t_0): parameter

K is a solution to time-depend Schrödinger Eq in the variables (x,t),

B.C. if $\psi(x,t_0) = \delta(x-x_0)$ $\Rightarrow$ at time $=t$, $\psi(x,t) = K(x,t;x_0,t_0)$

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|U(t,t_0)\overset{\mathbb{1}}{\downarrow}|\psi(t_0)\rangle$$

$$= \int dx_0 \, \underbrace{\langle x|U(t,t_0)|x_0\rangle}_{=k(x,t;\,x_0,t_0)} \underbrace{\langle x_0|\psi(t_0)\rangle}_{=\psi(x_0,t_0)}$$

$$= \int dx_0 \, k(x,t;\,x_0,t_0)\,\psi(x_0,t_0)$$

k is the kernel of the integral transform that converts an initial WF into a final one.

* The propagator for the free particle.

$$\hat{H} = \frac{\hat{p}^2}{2m}, \quad (\text{~~time-independent~~})$$

Set $t_0 = 0$, $\quad U(t) = U(t,0) = \exp\left(-\frac{iHt}{\hbar}\right)$

$$\Rightarrow k(x,x_0,t) = \langle x|\exp\left(-i\frac{\hat{p}^2 t}{2m\hbar}\right)\overset{\mathbb{1}}{\downarrow}|x_0\rangle$$

$$= \int dp \langle x|\exp\left(-\frac{i\hat{p}^2 t}{2m\hbar}\right)|p\rangle\langle p|x_0\rangle$$

$$= \int dp \, e^{-\frac{ip^2 t}{2m\hbar}} \langle x|p\rangle\langle p|x_0\rangle$$

Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

$$= \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}\underbrace{\left(p(x-x_0) - \frac{p^2 t}{2m}\right)}_{= -\frac{t}{2m}\left(p - \frac{m}{t}(x-x_0)\right)^2 + \frac{m}{2t}(x-x_0)^2}\right)$$

$$= \sqrt{\frac{m}{i2\pi\hbar t}} \exp\left[\frac{i}{\hbar}\frac{m(x-x_0)^2}{2t}\right]$$

for 3D $\quad k(\vec{x},\vec{x}_0,t) = \left(\frac{m}{2i\pi\hbar t}\right)^{\frac{3}{2}} \exp\left[\frac{i}{\hbar}\frac{m(\vec{x}-\vec{x}_0)^2}{2t}\right]$