Quantum Mechanics - I

Dec. 21. 2009

Theory of Angular Momentum

$\boxed{\text{Transformation and Invariance}}$

$$T\{|u\rangle\} = |u'\rangle$$

$\downarrow$

is not an operator on the Hilbert space yet

T : rotation, translation, a Lorentz boost spatial inversion etc ..

either
- ① transforming the system itself : active
- ② leave the system intact and transform to a new coordinate with observer O' : passive

• Physical law shall be ~~left~~ invariant.

say $A|\phi_i\rangle = \lambda_i|\phi_i\rangle$ $\qquad \text{Prob}(A=\lambda_i) = |\langle\phi_i|u\rangle|^2$

under T, $\quad T[|\phi_i\rangle] = |\phi_i'\rangle$

$\Rightarrow \; |\langle\phi_i|u\rangle|^2 = |\langle\phi_i'|u'\rangle|^2 \quad \Rightarrow$ must be some operator $\hat{T}$ on the Hilbert space corresponds to the actual transformation

$\Rightarrow \; \hat{T}|u\rangle = |u'\rangle \quad , \quad \hat{T}|\phi_i\rangle = |\phi_i'\rangle$

$|\langle\phi_i|u\rangle|^2 = |\langle\phi_i|\hat{T}^+\hat{T}|u\rangle|^2$

if $\hat{T}^+\hat{T} = \mathbb{I} \Rightarrow \hat{T}$ unitary

it is the case for those can be built from infinitesimal transformation : rotation, proper Lorentz boost, spatial displacement, time displacement, or $\mathbb{P}$ & $\mathbb{C}$

for $\hat{T}^+\hat{T} = -1$, $\Rightarrow \hat{T}$ anti-unitary.

$\mathbb{T}$ time reversal. (next semester)

☆ Unitary rotation operator R.

R in 3-dim Euclidean space forms a group. $\overset{\text{isomorphic}}{=}$ all real orthogonal 3×3 matrices with $\underline{\det = 1}$ $SO(3)$ group.

($\mathbb{I}$, Closure, Inverses, Associativity)

For each rotation $R \Rightarrow$ an $\underline{\text{operator}}$ on the Hilbert space.

these operators also form a group.

(but not necessarily isomorphic to $SO(3)$!!)

• Any rotation can be specified by

$$\cdots R_{\hat{n}_3}(\theta_3)\cdot R_{\hat{n}_2}(\theta_2)\cdot R_{\hat{n}_1}(\theta_1)$$

& the corresponding operators $\cdots \hat{R}_{\hat{n}_3}(\theta_3)\cdot\hat{R}_{\hat{n}_2}(\theta_2)\cdot\hat{R}_{\hat{n}_1}(\theta_1)$ on the Hilbert space

Let's now consider a rotation $R_{\hat{n}}(\epsilon)$, $\epsilon \ll 1$,

$$R_{\hat{n}}(\epsilon) = \mathbb{1}_{3\times3} - \frac{i}{\hbar}\epsilon J_{\hat{n}}$$

$J_{\hat{n}}$: the generator

Same dimension as $[\hbar] = [M\frac{L^2}{T}] = L \cdot \frac{ML}{T} = $ [angular momentum]

R : unitary

$$RR^{+} \simeq \mathbb{1} - \frac{i}{\hbar}\epsilon\,[J - J^{+}] = \mathbb{1}$$

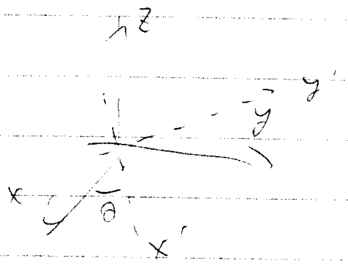
$\Rightarrow \quad J = J^{+}$: Hermitian

A finite rotation of amount θ about axis $\hat{n}$ can be built up from a succession of infinitesimal rotation, $\theta = N\epsilon$

$$R_{\hat{n}}(\theta) = \lim_{N\to\infty} \underbrace{R_{\hat{n}}(\epsilon) \cdots R_{\hat{n}}(\epsilon)}_{N} = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}\frac{\theta}{N}J_{\hat{n}}\right)^{N} = e^{\left(-\frac{i}{\hbar}\theta J_{\hat{n}}\right)}$$

Once this is defined, the entire theory of angular momentum in QM follows.

Consider a finite rotation in 2D



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=R_{\hat{z}}(\theta)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for $\theta \ll 1$

$$R_{\hat{z}}(\epsilon) \simeq \mathbb{1} + \begin{pmatrix} \cancel{\epsilon^2} & -\epsilon & 0 \\ \epsilon & \cancel{-\epsilon^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\epsilon^2)$$

Therefore $\frac{-iJ_z}{\hbar} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, similarly $\frac{-iJ_x}{\hbar} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\frac{-iJ_y}{\hbar} = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

and $$\left[\frac{J_x}{\hbar}, \frac{J_y}{\hbar}\right] = \left[i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, i\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\right] = \frac{i}{\hbar}J_z$$

you can easily show that $[J_i, J_j] = i\epsilon_{ijk}J_k\hbar$

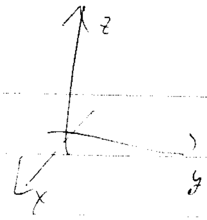
$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{321} = -\epsilon_{132} = +1$$

Properties of the angular momentum operators

define $J_{\pm} = J_x \pm iJ_y$. Then one has

$$[J_{\pm}, J_z] = [J_x \pm iJ_y, J_z] = i\hbar[-J_y \pm iJ_x] = \mp\hbar J_{\pm}$$

$$[J_z, J_{\pm}] = \pm\hbar J_{\pm}$$



More on the commutation relation.

☆ consider ~~the~~ a real sequence of 3-D rotation.

$$R_y(-\epsilon)\, R_x(-\epsilon)\, R_y(\epsilon)\, R_x(\epsilon)$$

$$= \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & -\epsilon \\ 0 & 1 & 0 \\ +\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & +\epsilon \\ 0 & -\epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix} \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & +\epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & +\epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{\epsilon^2}{2} & +\epsilon^2 & -\epsilon+\frac{\epsilon^3}{2} \\ 0 & 1-\frac{\epsilon^2}{2} & +\epsilon \\ +\epsilon & -\epsilon+\frac{\epsilon^3}{2} & 1-\epsilon^2+\frac{\epsilon^4}{4} \end{pmatrix} \cdot \begin{pmatrix} 1-\frac{\epsilon^2}{2} & +\epsilon^2 & +\epsilon-\frac{\epsilon^3}{2} \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & +\epsilon-\frac{\epsilon^3}{2} & 1-\epsilon^2+\frac{\epsilon^4}{4} \end{pmatrix} = \begin{pmatrix} 1-\frac{\epsilon^4}{2} & +\epsilon^2 & 0 \\ -\epsilon^2 & 1-\frac{\epsilon^4}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(\epsilon^3)$$

$$\simeq R_z(\epsilon^2)$$

☆ Therefore, the operators (or the transformation) on the Hilbert space must also respect this property: namely,

$$\exp\left(+\frac{i\epsilon}{\hbar}\hat{J}_y\right) \exp\left[+\frac{i\epsilon}{\hbar}\hat{J}_x\right] \exp\left[-\frac{i\epsilon}{\hbar}\hat{J}_y\right] \exp\left(-\frac{i\epsilon}{\hbar}\hat{J}_x\right] \simeq \exp\left[\frac{+i\epsilon^2}{\hbar}\hat{J}_z\right]$$

expand the LHS to $O(\epsilon^2)$.

$$\Rightarrow \left(\mathbb{1} + \frac{i\epsilon}{\hbar}\hat{J}_y - \frac{\epsilon^2}{2\hbar^2}\hat{J}_y^2\right)\left(\mathbb{1} + \frac{i\epsilon}{\hbar}\hat{J}_x - \frac{\epsilon^2}{2\hbar^2}\hat{J}_x^2\right)\left(\mathbb{1} - \frac{i\epsilon}{\hbar}\hat{J}_y - \frac{\epsilon^2}{2\hbar^2}\hat{J}_y^2\right)\left(\mathbb{1} - \frac{i\epsilon}{\hbar}\hat{J}_x - \frac{\epsilon^2}{2\hbar^2}\hat{J}_x^2\right)$$

Whatever representation

$$\simeq \left(\mathbb{1} + \frac{i\epsilon}{\hbar}(\hat{J}_y + \hat{J}_x) + \frac{\epsilon^2}{\hbar^2}\left(-\frac{\hat{J}_y^2}{2} - \frac{\hat{J}_x^2}{2} - \hat{J}_y\hat{J}_x\right)\right) \times \left[\mathbb{1} - \frac{i\epsilon}{\hbar}\left[\hat{J}_y + \hat{J}_x\right] + \frac{\epsilon^2}{2\hbar^2}\left(-\hat{J}_x^2 - \hat{J}_y^2 - \hat{J}_y\hat{J}_x\right)\right]$$

$$\simeq \mathbb{1} - \frac{\epsilon^2}{\hbar^2}\left(\hat{J}_x^2 + \hat{J}_y^2 + 2\hat{J}_y\hat{J}_x\right) + \frac{\epsilon^2}{\hbar^2}\left(\hat{J}_y^2 + \hat{J}_y\hat{J}_x + \hat{J}_x\hat{J}_y + \hat{J}_x^2\right) + O(\epsilon^3)$$

$$\simeq \mathbb{1} + \frac{\epsilon^2}{\hbar^2}\left[\hat{J}_x, \hat{J}_y\right] = \mathbb{1} + \frac{i\epsilon^2}{\hbar}\hat{J}_z$$

$$\Rightarrow \frac{1}{\hbar}\left[\hat{J}_x, \hat{J}_y\right] = +i\hat{J}_z \quad \text{or} \quad \left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z$$

And similarly, $\left[\hat{J}_i, \hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k$

$$J_+J_- - J_-J_+ = 2J_z$$

and $[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = -i(i\hbar J_z) - i(i\hbar J_z) = 2J_z$

Also, an important operator $J^2 = J_x^2 + J_y^2 + J_z^2$

$$= \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

$$= \frac{1}{2}J_-J_+ + \frac{1}{2}(J_-J_+ + 2J_z) + J_z^2 = J_-J_+ + J_z(J_z+1)$$

and $= \frac{1}{2}(J_+J_- - 2J_z) + \frac{1}{2}J_+J_- + J_z^2 = J_+J_- + J_z(J_z - 1)$

* The J^2 has only non-negative eigenvalues. Denote $|j\rangle$ the eigenstates of J^2. Then

$$\langle j|J^2|j\rangle = \langle j|J_xJ_x + J_yJ_y + J_zJ_z|j\rangle \quad , \text{ since } J_i \text{ is Hermitian.}$$

$$= \langle V_x|V_x\rangle + \langle V_y|V_y\rangle + \langle V_z|V_z\rangle \geq 0 \qquad |V_i\rangle = J_i|j\rangle$$

We will show later that $J^2|j\rangle = \hbar^2 j(j+1)|j\rangle$, $j \geq 0$.

* Furthermore, J^2 commutes with J_i.

$$[J_z, J^2] = \cancel{[J_z, J_z^2 + J_x^2 + J_y^2]} = \cancel{[J_z, J_xJ_x] + [J_z, J_yJ_y]}$$

$$= [J_z, J_+J_- + J_z(J_z-1)] = [J_z, J_+J_-] = J_zJ_+J_- - J_+J_zJ_- + J_+J_zJ_- - J_+J_-J_z$$

$$= [J_z, J_+]J_- + J_+[J_z, J_-] = +\hbar J_+J_- - \hbar J_+J_- = 0$$

Similarly $[J_x, J^2] = [J_x, J_y^2] + [J_x, J_z^2]$

$$= (J_xJ_y^2 - J_yJ_xJ_y + J_yJ_xJ_y - J_y^2J_x) + (J_xJ_z^2 - J_zJ_xJ_z + J_zJ_xJ_z - J_z^2J_x)$$

$$= [J_x, J_y]J_y + J_y[J_x, J_y] + [J_x, J_z]J_z + J_z[J_x, J_z]$$

$$= i\hbar J_zJ_y + i\hbar J_yJ_z - i\hbar J_yJ_z - i\hbar J_zJ_y = 0$$

and $[J_y, J^2] = 0$ (but $[J_i, J_j] = i\epsilon_{ijk}J_k\hbar$)

⇒ Therefore, we can form simultaneous eigenstates of J^2 and <u>one</u> of (J_x, J_y, J_z). It's ~~totally~~ arbitrary, usually people pick J_z.

⇒ $J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$, $J_z|j,m\rangle = \hbar m|j,m\rangle$

$$J_zJ_+|j,m\rangle = \hbar(J_+ + J_+J_z)|j,m\rangle = (m+1)\hbar\, \underline{J_+|j,m\rangle}$$

$$J_zJ_-|j,m\rangle = \hbar(-J_- + J_-J_z)|j,m\rangle = \hbar(m-1)\, \underline{J_-|j,m\rangle}$$

eigenstates of J_z

and since $[J_\pm, J^2] = [J_x \pm i J_z, J^2] = 0$

$J_+|j,m\rangle$ and $J_-|j,m\rangle$ are each eigenstates of J^2 with eigenvalue $j(j+1)$

Therefore $J_+|j\,m\rangle = C_+^{jm}\hbar|j, m+1\rangle$ where $C_\pm$ are consts.
$J_-|j\,m\rangle = C_-^{jm}\hbar|j, m-1\rangle$

from $J_-J_+|jm\rangle = [J^2 - J_z(J_z+1)]|jm\rangle = \hbar^2(j(j+1) - m(m+1))|jm\rangle$
and $J_+J_-|jm\rangle = [J^2 - J_z(J_z-1)]|jm\rangle = \hbar^2(j(j+1) - m(m-1))|jm\rangle$

So $\langle jm|J_-J_+|jm\rangle = |C_+^{jm}|^2 = j(j+1) - m(m+1)$ (recall that $(J_+)^\dagger = J_-$)
and $\langle jm|J_+J_-|jm\rangle = |C_-^{jm}|^2 = j(j+1) - m(m-1)$

$\Rightarrow$ $\cancel{L}$HS ≥ 0 unless $J_+|j,m\rangle$ or $J_-|jm\rangle = 0$

$\Rightarrow$ $j(j+1) \geq m(m+1)$ and $m(m-1) \leq j(j+1)$

But since one can increse / decrese (m) by applying J_+/J_- unless $m_{max} = j$ or $m_{min} = -j$ the above inequalities will not hold. Therefore, we ~~can~~ conclude that

$$m = \underbrace{-j, -j+1, -j+2 \cdots\cdots j-2, j-1, j}_{2j+1 \text{ values of } m}$$

This can only happen if j is integeral or half-integral:

$j = 0, \frac{1}{2}, 1, \frac{3}{2} \cdots$

scalar (meson), spinor (fermion), vector (meson photon, ..), graviton

say $j = 1.03$
$m = 1.03, 0.03, -0.97, -1.97 \cdots \to -\infty$
$j = 0.51$
$m = 0.51, -0.49, -1.49, \cdots -\infty$

(※) Now you see why the $J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$

(※) And, just from the geometry, plus the definition $\hat{R}(\theta) = e^{(-\frac{i}{\hbar}\theta \cdot \hat{J})}$, the entire theory of angular momentums in QM follows without any postulates!!

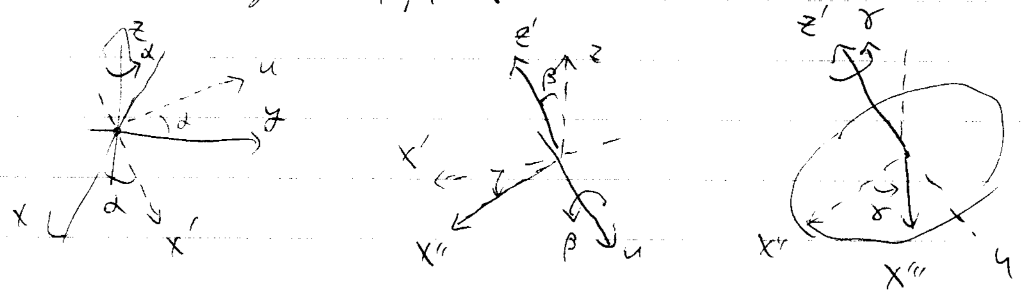
By convention, $C_{\pm}^{jm}$ are choosen to be real and positive.

$$\Rightarrow \quad C_{jm}^{+} = \sqrt{j(j+1) - m(m+1)} \qquad \& \qquad J_+ |jm\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\ |j, m+1\rangle$$

$$C_{jm}^{-} = \sqrt{j(j+1) - m(m-1)} \qquad\qquad J_- |j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\ |j, m-1\rangle$$

★ $\boxed{\text{Rotation matrices and representations}}$

Any rotation in 3-D can be decomposed into 3 succession rotations specified by the "Euler angles" α, β, γ.



The net rotation can be expressed as

$$R_{\hat{n}}(\theta) = R_{z'}(\gamma) \cdot R_u(\beta) \cdot R_z(\alpha)$$

However, for any matrix M, applying an orthogonal transformation matrix Q results in
$$M' = Q M Q^{-1}$$

We have $\quad R_{z'}(\gamma) = R_u(\beta) \cdot R_z(\gamma) \cdot R_u^{-1}(\beta)$

and $\quad R_u(\beta) = R_z(\alpha) \cdot R_y(\beta) \cdot R_z^{-1}(\alpha)$

Therefore
$$R_{\hat{n}}(\theta) = R_u(\beta) \cdot R_z(\gamma) \cdot R_u^{-1}(\beta)\ \overset{R_u(\beta)}{\cancel{R_z(\alpha) \cdot R_y(\beta) \cdot R_z^{-1}(\alpha)}} \cdot R_z(\alpha)$$
$$= R_u(\beta)\ R_z(\gamma)\ R_z(\alpha) \quad = R_z(\alpha) \cdot R_y(\beta) \cdot R_z^{-1}(\alpha)\ \ R_z(\alpha)\ R_z(\gamma)$$
$$\boxed{= R_z(\alpha) \cdot R_y(\beta) \cdot R_z(\gamma)}$$

For the corresponding operators on the Hilbert space,

$$\hat{R}_{\hat{n}}(\theta) = e^{-i\frac{\alpha}{\hbar}\hat{J}_z}\ e^{-\frac{i\beta}{\hbar}\hat{J}_y}\ e^{-\frac{i\gamma}{\hbar}\hat{J}_z}$$

Now consider a physical system, originally in the state $|j,m\rangle$.
Rotate the system, and the state vector becomes $R|j,m\rangle$.
It can be expanded in a complete set of angular eigenstates $|j',m'\rangle$

$$R|j,m\rangle = \sum_{j',m'} |j',m'\rangle\langle j',m'|R|j,m\rangle \quad , \text{ since } [R, J^2]=0 \Rightarrow \text{only } j'=j \text{ left.}$$

$$= \sum_{m'} |j,m'\rangle \langle j,m'|R|j,m\rangle$$

$$\equiv \sum_{m'} D^j_{m,m'}(\alpha,\beta,\gamma)\, |j,m'\rangle$$

and $D^j_{m,m'}(\alpha,\beta,\gamma) = \langle j\, m'|\, e^{-\frac{i}{\hbar}\alpha J_z}\, e^{-\frac{i\beta}{\hbar}J_y}\, e^{-\frac{i\gamma}{\hbar}J_z}\,|\, j\, m\rangle$

are elements of the so called "rotation matrix" which has dimension $(2j+1)\times(2j+1)$

$$\Rightarrow D^j_{m,m'}(\alpha,\beta,\gamma) = e^{-im'\alpha}\, e^{-i\gamma m}\, \underbrace{\langle j\,m'|\, e^{-\frac{i\beta}{\hbar}J_y}\,|\, j,m\rangle}_{\equiv\, d^j_{m,m'}(\beta)}$$

$j=\frac{1}{2}$ case

for $j=\frac{1}{2}$, $m=\pm\frac{1}{2}$. They are conveniently represented by the column spinors,

$$\psi(m=\tfrac{1}{2}) = \begin{pmatrix}1\\0\end{pmatrix} \qquad \psi(m=-\tfrac{1}{2}) = \begin{pmatrix}0\\1\end{pmatrix}$$

In this representation, it is obvious that

$$J_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar}{2}\sigma_z$$

Also,

$$J_+\begin{pmatrix}0\\1\end{pmatrix} = \hbar\begin{pmatrix}1\\0\end{pmatrix} , \quad J_+\begin{pmatrix}1\\0\end{pmatrix} = 0 \qquad \Rightarrow \quad J_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$J_-\begin{pmatrix}0\\1\end{pmatrix} = 0 , \quad J_-\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \qquad J_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$\Rightarrow$ from which it follows that

$$J_x = \frac{\hbar}{2}(J_+ + J_-) = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$J_y = \frac{-i}{2}(J_+ - J_-) = -\frac{i}{2}\hbar\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

Therefore $\exp\left(-\frac{i\beta}{\hbar}J_y\right) = \exp\left(-\frac{i\beta\hbar}{2\hbar}\sigma_y\right) = \exp\left(-\frac{i\beta\sigma_y}{2}\right)$

$$= \mathbb{1}_{2\times 2}\cdot\cos\frac{\beta}{2} - i\,\sigma_y \sin\frac{\beta}{2}$$

$$= \begin{array}{c} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \\ m=\tfrac{1}{2} \qquad m=-\tfrac{1}{2} \end{array} = d^{j}_{m,m'}(\beta) \text{ in this representation.}$$

And
$$D^{j}_{m,m'}(\alpha,\beta,\gamma) = \begin{array}{c} m'=\tfrac{1}{2} \\ m'=-\tfrac{1}{2} \end{array}\begin{pmatrix} e^{-\frac{i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2}, & -e^{\frac{i(\gamma-\alpha)}{2}}\sin\frac{\beta}{2} \\ +e^{\frac{i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2}, & e^{+\frac{i(\alpha+\beta)}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

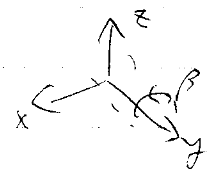
$\det\left(D^{j}_{m,m'}\right) = +1$, The set of all such matrices for all possible (α,β,γ) forms the continuous group $SU(2)$.

special, unitary, 2×2
$\Rightarrow \det = 1$

It's ~~one of~~ the most simple example of Lie group. (19th Norwegian mathematician Sophus Lie)

$SU(2)$ is closely related to $SO(3)$, but not 1-1 (isomorphism) because $\left(\frac{\beta}{2}\right)$. $\Rightarrow$ needs 4π to return to the original state.

Say, $\psi = \begin{pmatrix}1\\0\end{pmatrix}$, apply the rotation along $\hat{y}$ by β
spin up



$\Rightarrow \psi' = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}$,

$\beta = \frac{\pi}{2}$, spin $+\hat{x}$ $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

$\beta = \pi$ spin $-\hat{z}$: $\begin{pmatrix}0\\1\end{pmatrix}$

$\beta = \frac{3}{2}\pi$, spin $-\hat{x}$: $\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ +\frac{1}{\sqrt{2}}\end{pmatrix}$

$\beta = 2\pi$ spin along $+\hat{z}$: $\begin{pmatrix}-1\\0\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix}$, $\neq \psi$

$\boxed{j=1 \text{ case}}$ The 3 states $m=+1,0,-1$, are represented by

$$\psi(m=+1)=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \psi(m=0)=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \psi(m=-1)=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Clearly, $J_z=\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ also $J_+=\hbar\begin{pmatrix}0&\sqrt{2}&0\\0&0&\sqrt{2}\\0&0&0\end{pmatrix}$, $J_-=\hbar\begin{pmatrix}0&0&0\\\sqrt{2}&0&0\\0&\sqrt{2}&0\end{pmatrix}$

(from $C^{\pm}_{jm}$)

$\Rightarrow$ from the above, it follows that

$$J_x=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad J_y=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}=\hbar\,\mathcal{Y}_y$$

Again we shall work out $\exp\left(-\frac{i\beta J_y}{\hbar}\right)=\mathbb{1}_{3\times3}-i\beta\,\mathcal{Y}_y+\frac{(-i)^2}{2!}(\beta)^2(\mathcal{Y}_y)^2+\cdots$

$$(\mathcal{Y}_y)^2=\frac{1}{2}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}$$

$$(\mathcal{Y}_y)^3=\frac{1}{2\sqrt{2}}\begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}=\frac{1}{2\sqrt{2}}\begin{pmatrix}0&-2i&0\\2i&0&-2i\\0&2i&0\end{pmatrix}=\mathcal{Y}_y$$

$$\Rightarrow \exp\left(-\frac{i\beta J_y}{\hbar}\right)=\mathbb{1}-i\left(\beta-\frac{\beta^3}{3!}+\cdots\right)\mathcal{Y}_y+\left[-\frac{\beta^2}{2!}+\frac{\beta^4}{4!}-\cdots\right]\mathcal{Y}_y^2$$

$$=\mathbb{1}_{3\times3}-i\sin\beta\,\mathcal{Y}_y+(\cos\beta-1)\,\mathcal{Y}_y^2$$

then

$$d^1_{m'm}(\beta)=\begin{array}{c|ccc} & 1 & 0 & -1\\ \hline 1 & \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2}\\ 0 & \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}}\\ -1 & \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2}\end{array},\quad \left(D=e^{-im'\alpha}e^{im\gamma}d\right)$$

From this, if a particle with spin-1, originally pointing to $\hat{z}$, $\psi(m=+1)=\begin{pmatrix}1\\0\\0\end{pmatrix}$

then a rotation about $\hat{y}$ by angle β results in the new spin-function

$$\psi'=\begin{pmatrix}\frac{1+\cos\beta}{2}\\ \frac{1}{\sqrt{2}}\sin\beta\\ \frac{1-\cos\beta}{2}\end{pmatrix}\qquad \beta=\frac{\pi}{2},\ +\hat{x}:\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix},\quad \beta=\pi,\ -\hat{z}:\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\beta=\frac{3}{2}\pi,\ -\hat{x}:\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{2}\end{pmatrix},\quad \beta=2\pi,\ +\hat{z}:\begin{pmatrix}1\\0\\0\end{pmatrix}=\psi(m=+1)$$

in Hilbert space

$\Rightarrow$ for all integer j's, the rotation matrices D and the rotation matrices in $SO(3)$ is 1 to 1

meanwhile, for all half-integral j, the correspondence is 2 to 1

* In this class, I'd rather emphasis physics instead the mathematical tools. For those who want to ~~study~~ know more about the group theory, a very good ref. is

Lie Algebra in particle physics
by Howard Georgi