Orbital angular momentum

- $\vec{L} = \vec{r} \times \vec{p}$ $(\vec{L})_k = \epsilon_{kij}\, x_i p_j$
- In QM $[x_i, p_j] = i\hbar\,\delta_{ij}$

(a)
$$[X_i, L_j] = [X_i, \epsilon_{jab} X_a P_b] = \epsilon_{jab}[X_i, X_a P_b]$$
$$= \epsilon_{jab}(X_i X_a P_b - \cancel{\cdots} - X_a P_b X_i)$$
$$= \epsilon_{jab}(X_a [X_i, P_b]) = i\hbar\, \epsilon_{jab} X_a \delta_{ib} = i\hbar\, \epsilon_{aij} X_a$$

(b) and
$$[P_i, L_j] = \cancel{\epsilon_{iab} X_a P_b}\, [P_i, \epsilon_{jab} X_a P_b]$$
$$= \epsilon_{jab}(P_i X_a P_b - X_a P_b P_i) = \epsilon_{jab}[P_i, X_a] P_b$$
$$= -i\hbar\, \delta_{ia} \epsilon_{jab} P_b = i\hbar\, \epsilon_{bij} P_b$$

(c)
$$[L_i, L_j] = \cancel{[\epsilon_{iab} X_a P_b, L_j] = \epsilon_{iab}(X_a P_b L_j - P_b L_j X_a)}$$
$$\cancel{= \epsilon_{iab}(X_a P_b L_j - X_a L_j P_b + X_a L_j P_b - P_b L_j X_a}$$
$$= [\epsilon_{iab} X_a P_b, \epsilon_{jcd} X_c P_d] = \epsilon_{iab}\epsilon_{jcd}[X_a P_b, X_c P_d]$$
$$= \epsilon_{iab}\epsilon_{jcd}(X_a P_b X_c P_d - X_c P_d X_a P_b)$$
$$= \epsilon_{iab}\epsilon_{jcd}\left(X_a(\cancel{X_c P_b} - i\hbar\delta_{bc}) P_d - X_c(\cancel{X_a P_d} - i\hbar\delta_{ad}) P_b \right)$$
$$= i\hbar\, \epsilon_{iab}\epsilon_{jcd}(-\delta_{bc} X_a P_d + \delta_{ad} X_c P_b)$$
$$= i\hbar(-\epsilon_{iab}\epsilon_{jbd} X_a P_d + \epsilon_{iab}\epsilon_{jca} X_c P_b)$$
$$= i\hbar\left(-(\delta_{di}\delta_{ja} - \delta_{da}\cancel{\delta_{ji}}) X_a P_d + (\delta_{bj}\delta_{ic} - \delta_{bc}\cancel{\delta_{ij}}) X_c P_b \right)$$
$$= i\hbar(X_i P_j - X_j P_i) = i\hbar\, \epsilon_{ijk}(\epsilon_{kab} X_a P_b)$$
$$\Rightarrow \quad [L_i, L_j] = i\hbar\, \epsilon_{ijk} L_k$$

☆ Therefore, from the general discussion of QM angular momentum, it follows that

$$L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$$
$$L_z |l, m\rangle = \hbar m |l, m\rangle$$
$$L_\pm |l, m\rangle = \sqrt{l(l+1) - m(m\pm1)}\; |l, m\pm1\rangle$$

However, there is an important restriction: l can only be integer!!

To see this, let

$$a = \frac{x + i p_x}{\sqrt{2}}, \quad a^+ = \frac{x - i p_x}{\sqrt{2}}, \quad \text{then} \quad [a, a^+] = \hbar$$

$$b = \frac{y + i p_y}{\sqrt{2}}, \quad b^+ = \frac{y - i p_y}{\sqrt{2}}, \quad [b, b^+] = \hbar$$

and $[a, b^+] = [a, b] = 0$

also, L_z can be expressed as

$$i L_z = a^+ b - b^+ a = \frac{1}{\sqrt{2}}(x - i p_x)\frac{1}{\sqrt{2}}(y + i p_y) - \frac{1}{\sqrt{2}}(y - i p_y)\frac{1}{\sqrt{2}}(x + i p_x)$$

$$= -\frac{i}{2} y p_x + \frac{i}{2} x p_y + \frac{i}{2} x p_y - \frac{i}{2} y p_x$$

define $d = \frac{a - i b}{\sqrt{2}}$, $c = \frac{a + i b}{\sqrt{2}}$ $\Rightarrow$ $a = \frac{1}{\sqrt{2}}(c + d)$, $b = \frac{-i}{\sqrt{2}}(c - d)$

then

$$[c, c^+] = \frac{1}{2}[a + i b, a^+ - i b^+] = \hbar$$

$$[d, d^+] = \hbar$$

$$[c, d] = \frac{1}{2}(a + i b, a - i b) = 0 = \frac{1}{2}[a + i b, a^+ + i b^+] = [c, d^+]$$

So, L_z can be written as

$$L_z = -i(a^+ b - b^+ a) = d^+ d - c^+ c$$

from our study of SHO, we know that $d^+ d$ & $c^+ c$ have zero or positive integer eigenvalues (in the unit of $\hbar$) $\Rightarrow$ m must be integral. ($\hbar$)

$\Rightarrow$ l must also be integral ($\hbar$)

Usually, people use spherical polar coordinates.

$x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, $r = \sqrt{x^2 + y^2 + z^2}$

$\tan\phi = \frac{y}{x}$

$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$

$\frac{d\phi}{c^2\phi} = -\frac{y}{x^2} dx$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{x^2} c^2\phi \frac{\partial}{\partial \phi} + \frac{z x}{r^3 \sin\theta}\frac{\partial}{\partial \theta}$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$$

$$= \frac{y}{r}\frac{\partial}{\partial r} + \frac{\cos^2\phi}{x}\frac{\partial}{\partial \phi} + \frac{yz}{r^3 \sin\theta}\frac{\partial}{\partial \theta} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\sin\phi\, c\theta}{r}\frac{\partial}{\partial\theta}$$

$$L_z = x p_y - y p_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -i\hbar\left[r s\theta c\phi\left(\cancel{s\theta s\phi\frac{\partial}{\partial r}} + \frac{c\phi}{r s\theta}\frac{\partial}{\partial\phi} + \cancel{\frac{s\phi c\theta}{r}\frac{\partial}{\partial\theta}}\right) - r s\theta s\phi\left(\cancel{s\theta c\phi\frac{\partial}{\partial r}} - \frac{s\phi}{r s\theta}\frac{\partial}{\partial\phi} + \cancel{\frac{c\phi c\theta}{r}\frac{\partial}{\partial\theta}}\right)\right]$$

$$\boxed{L_z = -i\hbar\frac{\partial}{\partial\phi}}$$

since $L_z|l,m\rangle = m\hbar|l,m\rangle$

$$\langle x| -i\hbar\frac{\partial}{\partial\phi}|l,m\rangle = m\hbar\langle x|l,m\rangle \Rightarrow \langle x|l,m\rangle = f_{lm}(\theta)e^{+im\phi}$$

$$\psi_{lm} = f_{lm}(\theta)e^{+im\phi}$$

Now since $L^2 = L_-L_+ + L_z(L_z+\hbar)$

and $L_\pm = L_x \pm iL_y$

$$= (y p_z - z p_y) \pm i(z p_x - x p_z)$$

$$= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \pm \hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \cancel{\frac{\partial\phi}{\partial z}}^{\,0}\frac{\partial}{\partial\phi} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}$$

$$= \frac{z}{r}\frac{\partial}{\partial r} - \frac{1}{s\theta}\left(\frac{1}{r} - \frac{z^2}{r^3}\right)\frac{\partial}{\partial\theta}$$

$$= c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial\theta}$$

$$= -i\hbar\left[r s\theta c\phi\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial\theta}\right) - r c\theta\left(s\theta s\phi\frac{\partial}{\partial r} + \frac{c\phi}{r s\theta}\frac{\partial}{\partial\phi} + \frac{s\phi c\theta}{r}\frac{\partial}{\partial\theta}\right)\right]$$
$$\pm\hbar\left[r c\theta\left(s\theta c\phi\frac{\partial}{\partial r} - \frac{s\phi}{r s\theta}\frac{\partial}{\partial\phi} + \frac{c\phi c\theta}{r}\frac{\partial}{\partial\theta}\right) - r s\theta s\phi\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial\theta}\right)\right]$$

$$= \hbar(\mp x - iy)\frac{\partial}{\partial z} + \hbar z\left(i\frac{\partial}{\partial y} \pm \frac{\partial}{\partial x}\right)$$

$$= \mp\hbar e^{\pm i\phi}\sin\theta\, r\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial\theta}\right) + \hbar r c\theta\left(i\left(s\theta s\phi\frac{\partial}{\partial r} + \frac{c\phi}{r s\theta}\frac{\partial}{\partial\phi} + \frac{s\phi c\theta}{r}\frac{\partial}{\partial\theta}\right) \pm \left(s\theta c\phi\frac{\partial}{\partial r} - \frac{s\phi}{r s\theta}\frac{\partial}{\partial\phi} + \frac{c\phi c\theta}{r}\frac{\partial}{\partial\theta}\right)\right)$$

$$= \mp\hbar e^{\pm i\phi} s\theta\, r\left(c\theta\frac{\partial}{\partial r} - \frac{s\theta}{r}\frac{\partial}{\partial\theta}\right) + \hbar r c\theta\left(\pm s\theta e^{\pm i\phi}\frac{\partial}{\partial r} + i\frac{e^{\pm i\phi}}{r s\theta}\frac{\partial}{\partial\phi} \pm \frac{c\theta}{r}e^{\pm i\phi}\frac{\partial}{\partial\theta}\right)$$

$$\boxed{L_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\frac{\partial}{\partial\phi}\right)}$$

Thus, $L^2 = \hbar^2 e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\frac{\partial}{\partial\phi}\right)\left[e^{+i\phi}\left(+\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\frac{\partial}{\partial\phi}\right)\right] - i\hbar\frac{\partial}{\partial\phi}\left(-i\hbar\frac{\partial}{\partial\phi} + 1\right)$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\Rightarrow \quad L^2 \psi_{\ell m} = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right) \psi_{\ell m} = \hbar^2 \ell(\ell+1)\, \psi_{\ell m}$$

$$\psi_{\ell m} = f_{\ell m}(\theta)\, e^{+im\phi}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \ell(\ell+1) - \frac{m^2}{\sin^2\theta} \right) f_{\ell m}(\theta) = 0$$

The solution that is analytic for all θ is the associated Legendre polynomial P_ℓ^m

$$\Rightarrow \quad \psi_{\ell m} = \underbrace{N_{\ell m}}_{\text{normalization}} \times P_\ell^m(\theta) \times \exp(im\phi)$$

with a standard choice of $N_{\ell m}$, we have the spherical harmonics.

$$\Rightarrow \quad Y_\ell^m(\theta,\phi) = \sqrt{\left(\frac{2\ell+1}{4\pi}\right)\frac{(\ell-m)!}{(\ell+m)!}}\; \frac{1}{2^\ell \ell!}\, e^{im\phi} (-\sin\theta)^m \left(\frac{d}{d\cos\theta}\right)^{\ell+m} (\cos^2\theta - 1)^\ell$$

defined for $m \geq 0$, and $Y_\ell^{-m}(\theta,\phi) = (-1)^m\, Y_\ell^{m*}(\theta,\phi)$

$$\int Y_{\ell'}^{m'*}\, Y_\ell^m \; d(\cos\theta)\, d(\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}, \quad Y_2^{-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi}$$

$$Y_2^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{+2i\phi}, \quad Y_2^{-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\phi}$$

$$\langle \hat{n} | L_- | l, -l \rangle = 0$$

$$= \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\frac{\partial}{\partial\phi}\right) f_{l,-l}(\theta) e^{-il\phi} = 0$$

$$\Rightarrow \quad -f'_{l,-l} + l\frac{c\theta}{s\theta} f_{l,-l} = 0$$

$$\frac{d f'}{f} = l\frac{c\theta}{s\theta} d\theta$$

$$\Rightarrow \quad d(\ln f) = l\, d(\ln s\theta) \quad \Rightarrow \quad f_{l,-l}(\theta) = N \sin^l\theta$$

$$\Rightarrow \quad Y_l^{-l}(\theta,\phi) = N \sin^l\theta\, e^{-il\phi}$$

The normalization const is determined by requiring that

$$1 = \int d\phi\, d(\cos\theta)\, |N|^2 \sin^{2l}\theta = |N|^2 2\pi \int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)^l = 2\pi|N|^2\int_{-1}^{1} dx\,(1-x^2)^l$$

$$= 2\pi|N|^2 \int_{-1}^{1} dx\,(1+x)^l(1-x)^l$$

change variable $t = \frac{1+x}{2}$, $dt = \frac{1}{2}dx$

$$= 2\pi|N|^2\int_0^1 2dt\,(2t)^l(2(1-t))^l = 2\pi|N|^2 2^{(2l+1)}\int_0^1 dt\, t^l(1-t)^l$$

The last ~~expression~~ integral is Beta function

$$B(p,q) = \int_0^1 dt\, t^{p-1}(1-t)^{q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad \Gamma(p) = (p-1)!$$

$$\Rightarrow \quad 1 = 2\pi|N|^2 2^{(2l+1)}\frac{\Gamma(l+1)\Gamma(l+1)}{\Gamma(2l+2)} = 2\pi|N|^2 2^{(2l+1)}\frac{(l!)^2}{(2l+1)!}$$

or $|N| = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}$, by convention, we choose N to be real

$$\Rightarrow \quad \boxed{Y_l^{-l}(\theta,\phi) = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sin^l\theta\, e^{-il\phi}}$$

$$\langle\hat{n}| L_+ |l,m\rangle = \hbar\langle\hat{n}|\sqrt{l(l+1)-m(m+1)}\,|l,m+1\rangle$$

$$= \hbar\langle\hat{n}|\sqrt{(l-m)(l+m+1)}\,|l,m+1\rangle$$

Then we find

$$Y_\ell^m(\theta,\phi) = \frac{1}{\sqrt{(\ell-m+1)(\ell+m)}} e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\frac{\partial}{\partial\phi}\right) Y_\ell^{m-1}(\theta,\phi)$$

$$= \frac{1}{\sqrt{(\ell-m+1)(\ell+m)}}\frac{1}{\sqrt{(\ell+m-1)(\ell-m+2)}} \cdots\cdots \frac{1}{\sqrt{(1)(\ell+\ell)}}\left[e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\frac{\partial}{\partial\phi}\right)\right]^{\ell+m} Y_\ell^{-\ell}(\theta,\phi)$$

$$= \sqrt{\frac{(\ell-m)!}{(2\ell)!(\ell+m)!}}\left[\underbrace{e^{i\phi}}_{\text{increase by } +1}\left(\frac{\partial}{\partial\theta} + i\frac{c\theta}{s\theta}\underbrace{\frac{\partial}{\partial\phi}}_{\text{give eigenvalue } im}\right)\right]^{\ell+m} Y_\ell^{-\ell}(\theta,\phi)$$

$$= \frac{1}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\sqrt{\frac{(\ell-m)!}{(2\ell)!(\ell+m)!}}\, e^{im\phi}\left(\frac{d}{d\theta} - (m-1)\frac{c\theta}{s\theta}\right)\cdot\left(\frac{d}{d\theta} - (m-2)\frac{c\theta}{s\theta}\right)\cdots$$

$$\cdots\left(\frac{d}{d\theta} + (\ell-1)\frac{c\theta}{s\theta}\right)\left[\frac{d}{d\theta} + \ell\frac{c\theta}{s\theta}\right]\sin^\ell\theta$$

Now applying the trick that

$$\left(\frac{d}{d\theta} + k\frac{c\theta}{s\theta}\right) = \frac{1}{\sin^k\theta}\frac{d}{d\theta}\sin^k\theta = \frac{k\sin^{k-1}\theta\cos\theta}{\sin^k\theta} + \frac{d}{d\theta}$$

$$\Rightarrow Y_\ell^m(\theta,\phi) = \frac{1}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\, e^{im\phi}\left(\frac{1}{\sin^{1-m}\theta}\frac{d}{d\theta}\sin^{1-m}\theta\right)\cdot\left(\frac{1}{\sin^{2-m}\theta}\frac{d}{d\theta}\sin^{2-m}\theta\right)\cdots$$

$$\cdots\left(\frac{1}{\sin^{\ell-1}\theta}\frac{d}{d\theta}\sin^{\ell-1}\theta\right)\left(\frac{1}{\sin^\ell\theta}\frac{d}{d\theta}\sin^\ell\theta\right)\sin^\ell\theta$$

$$= \frac{1}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\, e^{im\phi}\sin^m\theta\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^{\ell+m}\sin^{2\ell}\theta$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx}$

$$\boxed{Y_\ell^m(\theta,\phi) = \frac{(-1)^{\ell+m}}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\, e^{im\phi}(1-x^2)^{\frac{m}{2}}\left(\frac{d}{dx}\right)^{\ell+m}(1-x^2)^\ell}$$

works for $m>0$

and $Y_\ell^{-m}(\theta,\phi) = (-1)^m Y_\ell^{m*}(\theta,\varphi)$