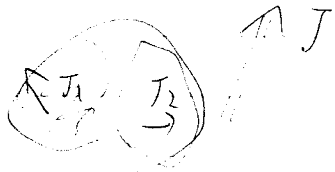
Addition of angular Momentum, Clebsch-Gordon coefficients

consider a system with angular momentum J which consists of 2 parts with J_1 & J_2,



$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

We assume that
$$[J_{1i}, J_{1j}] = i\epsilon_{ijk}\hbar J_{1k}$$
$$[J_{2i}, J_{2j}] = i\epsilon_{ijk}\hbar J_{2k}$$

also $[J_{1i}, J_{2j}] = 0$ $\Rightarrow$ each component of the angular momentum of one subsystem may be measured with disturbing any component of the other subsystem

$\Rightarrow$ There are states which are simultaneous eigenstates of J_1^2, J_{1z} ; J_2^2, J_{2z}, denoted as

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle \quad \text{or} \quad |j_1 m_1 ; j_2 m_2\rangle$$

On the other hand, eigenstate $|j\,m\rangle$ of J^2, J_z can be expressed as linear combinations of $|j_1 m_1\rangle |j_2 m_2\rangle$ for a given $\underline{j_1 \& j_2}$

$$|j\,m\rangle = \sum_{m_1 m_2} |j_1 m_1 ; j_2 m_2\rangle \underbrace{\langle j_1 m_1 ; j_2 m_2 | j\,m\rangle}$$

These coeff's are elements of a unitary transform matrix, and are called "Clebsch-Gordon" coeff.

Properties of CG coeff.

(a). $J_z = J_{1z} + J_{2z}$ acts on the above eq.

$$0 = \sum_{m_1 m_2} (m - m_1 - m_2) \underbrace{|j_1 m_1 ; j_2 m_2\rangle}_{\text{linearly indepent for different } m_1 m_2} \langle j_1 m_1 ; j_2 m_2 | j\,m\rangle$$

$$\Rightarrow \quad (m - m_1 - m_2) \langle j_1 m_1 ; j_2 m_2 | j\,m\rangle = 0$$

If $\langle j_1 m_1 ; j_2 m_2 | j\,m\rangle \neq 0 \Rightarrow \boxed{m = m_1 + m_2} \Rightarrow$ sum over only one variable

$$\Rightarrow \quad |j, m\rangle = \sum_{m_1} |j_1 m_1 ; j_2, (m - m_1)\rangle \langle j_1 m_1 ; j_2, (m - m_1) | j, m\rangle$$

(1-a)

2 possible choices of simultaneous eigenbasis

$[J_{1z}, J_1^2] = 0$
$[J_{2z}, J_2^2] = 0$
$\forall$ $[J_{1i}, J_{1j}] = i\epsilon_{ijk} J_{1k}$
$[J_{2i}, J_{2j}] = i\epsilon_{ijk} J_{2k}$
$[J_{1i}, J_{2j}] = 0$

(a) $J_1^2,\ J_2^2,\ J_{1z},\ J_{2z}$ $\Rightarrow$ This is obvious

(b). $J^2,\ J_1^2,\ J_2^2,\ J_z\ (= J_{1z} + J_{2z})$

$[J^2, J_z] = 0$, follows because $[J_i, J_j] = [J_{1i} + J_{2i}, J_{1j} + J_{2j}]$

$$= [J_{1i}, J_{1j}] + [J_{2i}, J_{2j}] = i\epsilon_{ijk}(J_{1k} + J_{2k})$$
$$= i\epsilon_{ijk} J_k$$

$$[J_1^2, J_z] = [J_{1z}^2 + J_{1x}^2 + J_{1y}^2, J_{1z} + J_{2z}] = [J_1^2, J_{1z}] = 0$$

$$[J_2^2, J_z] = [J_{2x}^2 + J_{2y}^2 + J_{2z}^2, J_{1z} + J_{2z}] = [J_2^2, J_{2z}] = 0$$

$$[J^2, J_1^2] = [J_1^2 + J_2^2 + 2 J_1 \cdot J_2, J_1^2] = 2[J_1 \cdot J_2, J_1^2] = 2[J_{1x} J_{2x} + J_{1y} J_{2y}, J_1^2]$$
$$= 2 J_{2x}[J_{1x}, J_1^2] + 2 J_{2y}[J_{1y}, J_1^2] = 0$$

$$\left(\text{or} = 2[J_{1z} J_{2z} + \tfrac{1}{2}(J_{1+} J_{2-} + J_{1-} J_{2+}), J_1^2] = J_{2-}[J_{1+}, J_1^2] + J_{2+}[J_{1-}, J_1^2] = 0 \right)$$

However, note that

$$[J^2, J_{1z}] = [J_1^2 + J_2^2 + 2 J_1 \cdot J_2, J_{1z}] = 2[J_{1z} J_{2z} + J_{1x} J_{2x} + J_{1y} J_{2y}, J_{1z}]$$
$$= 2 J_{2x}[J_{1x}, J_{1z}] + 2 J_{2y}[J_{1y}, J_{1z}]$$
$$= -2i J_{2x} J_{1y} + 2i J_{2y} J_{1x} \neq 0$$

~~*~~ print out the CG coeff table from PDG.

(b). Since $\langle j', m' | j, m\rangle = \delta_{j'j}\,\delta_{m'm}$, we have

$$\delta_{j'j}\,\delta_{m'm} = \sum_{m_1, n_1} \langle j' m' | j_1 m_1; j_2 (m'-m_1)\rangle \underbrace{\langle j_1 m_1; j_2 (m'-m_1) | j_1 n_1; j_2 (m-n_1)\rangle}_{\delta_{m_1 n_1} \times \delta_{m'-m_1, m-n_1} = \delta_{m_1 n_1}\delta_{mm'}} \langle j_1 n_1; j_2 (m-n_1) | j, m\rangle$$

Therefore

$$\boxed{\delta_{jj'} = \sum_{m_1} \langle j' m | j_1 m_1; j_2 (m-m_1)\rangle \langle j_1 m_1; j_2 (m-m_1) | j\, m\rangle}$$

Also, we will show later that all CG coeff. are <u>real</u>!

so, $\langle j', m | j_1 m_1; j_2 (m-m_1)\rangle = \langle j_1 m_1; j_2 (m-m_1) | j' m\rangle$

(c) Inverse relation: one can also expand the $|j_1 m_1; j_2 m_2\rangle$ in terms of the $|j, m\rangle$ basis, Then

$$|j_1 m_1; j_2 m_2\rangle = \sum_j |j, \overset{(m_1+m_2)}{m}\rangle \underline{\langle j\, m | j_1 m_1; j_2 m_2\rangle}$$

$\Rightarrow$ same CG coefficient.

(d) From the expression of (c).

$$\delta_{m_1' m_1}\,\delta_{m_2' m_2} = \langle j_1 m_1'; j_2 m_2' | j_1 m_1; j_2 m_2\rangle$$

$$= \sum_{j, j'} \langle j_1 m_1'; j_2 m_2' | j', (m_1'+m_2')\rangle \underbrace{\langle j', (m_1'+m_2') | j\, (m_1+m_2)\rangle}_{\delta_{j'j}\,\delta_{(m_1'+m_2'),(m_1+m_2)}} \langle j\, (m_1+m_2) | j_1 m_1; j_2 m_2\rangle$$

$$= \sum_j \langle j_1 m_1'; j_2 m_2' | j, m\rangle \langle j, m | j_1 m_1; j_2 m_2\rangle$$

$$\Rightarrow \boxed{\sum_j \langle j_1 m_1'; j_2 m_2' | j\, m\rangle \langle j\, m | j_1 m_1\, j_2 m_2\rangle = \delta_{m_1' m_1}\,\delta_{m_2'}\delta_{m_2}}$$

this is the second orthogonality relation.

(e). If $j_1 \geq j_2$, then the possible values of j are.

$j = j_1 + j_2,\ (j_1+j_2-1),\ (j_1+j_2-2),\ \cdots,\ (j_1 - j_2)$ ✻

① given j_2 & j_2, the maximum values of m_1 and m_2 are j_1 & j_2 respectively.

$\Rightarrow m_{max} = j_1 + j_2 \Rightarrow j_{max} = j_1 + j_2$

or $\boxed{|j_{max}, m_{max}\rangle = |j_1 j_1; j_2 j_2\rangle}$

In this case, the CG coeff. is $(+1)$.

Next, let's ~~consider~~ show why the minimum $j = j_1 - j_2$ (≥ 0)

for the $j = j_{max}$ case ~~state~~, there are $(2(j_1+j_2)+1)$ linear independent states

$j = j_{max} - 1$ " $(2(j_1+j_2-1)+1)$

⋮

$j = j_{min}$ " $(2j_{min}+1)$

$$\text{total} = 2\{j_{min} + (j_{min}+1) \cdots + (j_1+j_2)\} + \overbrace{(1+\cdots 1)}^{j_1+j_2-j_{min}+1}$$

$$= 2\left[j_{min}(j_1+j_2+1-j_{min}) + (0+1+\cdots+(j_1+j_2-j_{min}))\right] + (j_1+j_2-j_{min}+1)$$

$$= 2j_{min}(j_1+j_2+1-j_{min}) + \frac{2}{2}(j_1+j_2+1-j_{min})(j_1+j_2-j_{min}) + (j_1+j_2-j_{min}+1)$$

$$(2j_1+1)\times(2j_2+1) = (j_1+j_2-j_{min}+1)(2j_{min}+j_1+j_2-j_{min}+1) = (j_1+j_2+1)^2 - j_{min}^2$$

$$\Rightarrow j_{min}^2 = (j_1+j_2+1)^2 - (2j_1+1)(2j_2+1) = j_1^2 - 2j_1j_2 + j_2^2 = (j_1-j_2)^2$$

$$\Rightarrow \boxed{j_{min} = j_1 - j_2}$$

(+) The CG coeff. ① are real and ② can be easily worked out recursively.

We will demonstrate ~~these~~ points by working out several examples.

$\boxed{\text{example - I}}$ $\frac{1}{2}\times\frac{1}{2} \Rightarrow j = 1, 0$

$|j=1, m=1\rangle = |\uparrow\uparrow\rangle$

by applying the lowering operator $J_- = J_{1-} + J_{2-}$ on both sides.

$$\underbrace{\sqrt{1(1+1)-1(1-1)}}_{\sqrt{2}}\,|j=1, m=0\rangle = \underbrace{\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}}_{1}\,|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle$$

$$\Rightarrow |j=1, m=0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

Apply $\hat{J}_-$ again,

$$\sqrt{1(1+1)-0(0-1)}\;|j=1, m=-1\rangle = \frac{1}{\sqrt{2}}\Big[\underbrace{1}_{=\sqrt{\frac12(\frac12+1)-\frac12(\frac12-1)}}|\downarrow\downarrow\rangle + 1\,|\downarrow\downarrow\rangle\Big] \qquad : \frac{1}{\sqrt2},\ \frac{1}{\sqrt2}$$

$\Rightarrow \quad |j=1, m=-1\rangle = |\downarrow\downarrow\rangle \qquad\qquad \Rightarrow$ CG coeff. $= +1$

$j=1$ state is often called triplet state. $\Rightarrow$ $j=1$ state has exchange symmetry $(1 \leftrightarrow 2)$

☆ Now the $|j=0, m=0\rangle$ case, since $0=m=m_1+m_2$

This state must be a linear combination of $\uparrow\downarrow$ and $\downarrow\uparrow$

say $|0\,0\rangle = \alpha\,|\uparrow\downarrow\rangle + \beta\,|\downarrow\uparrow\rangle$

and we know $\langle 1,0|0,0\rangle = \frac{1}{\sqrt2}(\alpha+\beta) = 0$

also $\langle 0\,0|0\,0\rangle = |\alpha|^2+|\beta|^2 = 1$

$\Rightarrow \quad |0,0\rangle$ must be $\frac{1}{\sqrt2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ up to an arbitrary phase.

• $\boxed{j=0 \text{ state is antisymmetric under } 1\leftrightarrow 2 \text{ exchange}}$

This state is also called singlet state.

☆ the alternating exchange symmetry is a general property of CG coefficient when $j_1 = j_2$.

☆ explain how to read the CG coefficients table.

$\frac12 \times \frac12$

		1			
		+1			
$+\frac12$	$+\frac12$	1	1	0	
			0	0	
$|\uparrow\downarrow\rangle = +\frac12$	$-\frac12$		$\frac12$	$\frac12$	1
$|\downarrow\uparrow\rangle = -\frac12$	$+\frac12$		$\frac12$	$-\frac12$	-1
			$-\frac12$	$-\frac12$	1

• $\boxed{\text{example - 2}}$ $\quad \frac32 \times \frac12, \quad j = 2, 1.$

again start from $|2,2\rangle = |\frac32,+\frac32\rangle|\frac12,+\frac12\rangle$ or $|2,-2\rangle = |\frac32,-\frac32\rangle|\frac12,-\frac12\rangle$

$\downarrow$ by J_- $\qquad\qquad \downarrow$ by J_+

$$\sqrt{2(2+1)-2(2-1)}\,|2,1\rangle = \sqrt{\tfrac32(\tfrac32+1)-\tfrac32(\tfrac32-1)}\,|\tfrac32,+\tfrac12\rangle|\tfrac12,+\tfrac12\rangle + \sqrt{\tfrac12(\tfrac12+1)-\tfrac12(\tfrac12-1)}\,|\tfrac32,+\tfrac32\rangle|\tfrac12,-\tfrac12\rangle$$

$$\Rightarrow |2,1\rangle = \frac{\sqrt3}{2}|\tfrac32,\tfrac12\rangle|\tfrac12,\tfrac12\rangle + \frac12|\tfrac32,\tfrac32\rangle|\tfrac12,-\tfrac12\rangle$$

$$\cancel{4}\,|2,-1\rangle = \sqrt3\,|\tfrac32,-\tfrac12\rangle|\tfrac12,-\tfrac12\rangle + 1\,|\tfrac32,-\tfrac32\rangle|\tfrac12,+\tfrac12\rangle$$

$$\Rightarrow |2,-1\rangle = \frac{\sqrt3}{2}|\tfrac32,-\tfrac12\rangle|\tfrac12,-\tfrac12\rangle + \frac12|\tfrac32,-\tfrac32\rangle|\tfrac12,+\tfrac12\rangle$$

and $\sqrt{2(2+1)-1(1-1)}\;|2,0\rangle = \frac{\sqrt{3}}{2}\Big[\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\tfrac{3}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$

$\qquad + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$

$\qquad + \frac{1}{2}\Big[\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\;|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + 0\Big]$

$\Rightarrow \sqrt{6}\,|2,0\rangle = \frac{\sqrt{3}}{2}\Big[2\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\Big] + \frac{1}{2}\sqrt{3}\;|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

$= \sqrt{3}\;|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

or $|2,0\rangle = \frac{1}{\sqrt{2}}\Big(|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\Big)$

for $j=1$, let's consider $|j=1, m=1\rangle$ first, $1=m=(+\tfrac{3}{2})+(-\tfrac{1}{2})$ or $(+\tfrac{1}{2})+(\tfrac{1}{2})$

then $|1,1\rangle = \alpha\,|\tfrac{3}{2}\,\tfrac{3}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \beta\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,+\tfrac{1}{2}\rangle$

$|\alpha|^2+|\beta|^2=1$, and $\langle 2,1|1,1\rangle = \frac{\alpha}{2}+\frac{\sqrt{3}}{2}\beta = 0$

$\Rightarrow$ $|1,1\rangle$ can be choosen to be $\left(\alpha=\frac{\sqrt{3}}{2}\right.$ and $\left.\beta=-\frac{1}{2}\right)$ upto a phase

$|1,1\rangle = \frac{\sqrt{3}}{2}\,|\tfrac{3}{2}\,\tfrac{3}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \frac{1}{2}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,+\tfrac{1}{2}\rangle.$

Apply J_- on it, $\sqrt{1(1+1)-1(1-1)}\;|1,0\rangle = \frac{\sqrt{3}}{2}\Big(\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\;|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + 0\Big)$

$\qquad - \frac{1}{2}\Big\{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\tfrac{3}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2}+\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\;|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\Big\}$

$\Rightarrow \sqrt{2}\,|1,0\rangle = \frac{\sqrt{3}}{2}\sqrt{3}\;|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - \frac{1}{2}\Big(2\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + |\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\Big)$

$= +\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{3}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$

$\Rightarrow |1,0\rangle = \frac{1}{\sqrt{2}}\Big(|\tfrac{3}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{3}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\Big)$

$\vdots$ etc. You see how easy to calculate the CG coeff.

For any # of angular momentum sum. $J = J_1+J_2+\cdots J_N$

You can always start from $|j_1+j_2+j_3\cdots j_N,\; m=\sum_{i=1}^{N} j_i\rangle = |j_1\,j_1\rangle\,|j_2\,j_2\rangle\cdots|j_N\,j_N\rangle$

and apply the $\hat{J}_- = \hat{J}_{1-}+\hat{J}_{2-}+\cdots\hat{J}_{N-}$ to obtain any state you want.

(Better write a computer program to do that!)

(g) There is a general formula for the CG coefficients which has been derived by Wigner, Schwinger, Racah ... but too damned complicated to be useful in the normal circumstances!

$$\langle j_1 m_1 j_2 m_2 | j m\rangle = \delta_{m_1+m_2,m}\left[\frac{(2j+1)(j_1+j_2-j)!\,(j_1-j_2+j)!\,(-j_1+j_2+j)!}{(j_1+j_2+j+1)!}\right]^{\frac{1}{2}}$$

$$\times \sum_n (-1)^n \frac{[(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!]^{\frac{1}{2}}}{n!\,(j_1+j_2-j-n)!\,(j_1-m_1-n)!\,(j_2+m-n)!\,(j-j_2+m_1-n)!\,(j-j_2+m_1-n)!}$$

Sum over n which makes all the factorial argument positive.

Math vs. phys.

(h). There are some interesting symmetries among the CG coefficients:

$$\langle j_1 m_1\, j_2 m_2 | j_3 m_3\rangle = (-1)^{j_2+m_2}\left(\frac{2j_3+1}{2j_1+1}\right)^{\frac{1}{2}}\langle j_2\, -m_2\; j_3\, m_3 | j_1\, m_1\rangle$$

$$\langle j_1 m_1\, j_2 m_2 | j_3 m_3\rangle = (-1)^{j_1-m_1}\left(\frac{2j_3+1}{2j_2+1}\right)^{\frac{1}{2}}\langle j_3, m_3, j_1, -m_1 | j_2\, m_2\rangle$$

$$= (-1)^{j_1+j_2-j_3}\langle j_1\, -m_1, j_2, -m_2 | j_3, -m_3\rangle$$

(i) History: "3j" symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_2 & m_2 & m_3 \end{pmatrix} \equiv (-1)^{j_1-j_2-m_3}\sqrt{\frac{1}{2j_3+1}}\,\langle j_1 m_1 j_2 m_2 | j_3, -m_3\rangle$$