Quantum Mechanics-2 HW#4
Due 9:00am, April 21, 2010 (Wed).

No late HW will be accepted. So turn in whatever you have done.

1. (25%) (fine structure and Dirac equation) Dirac equation gives an exact fine-structure formula for hydrogen,

$$E_{nj} = mc^2 \left\{ \left[1 + \left(\frac{\alpha}{n - (j+1/2) + \sqrt{(j+1/2)^2 - \alpha^2}} \right)^2 \right]^{-1/2} - 1 \right\}$$

Expand it to order α^4, and show that you recover the result we have obtained in the class:

$$E_{nj} = -\frac{13.6\text{eV}}{n^2} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j+1/2} - \frac{3}{4} \right) \right]$$

2. (25%) (a) Let $\vec{a}, \vec{b}$ be two constant vectors. Show that

$$\int (\vec{a} \cdot \hat{r})(\vec{b} \cdot \hat{r}) \sin\theta \, d\theta \, d\phi = \frac{4\pi}{3} (\vec{a} \cdot \vec{b})$$

(b) Use this result to demonstrate that

$$\left\langle \frac{3(\vec{S}_p \cdot \hat{r})(\vec{S}_e \cdot \hat{r}) - \vec{S}_P \cdot \vec{S}_e}{r^3} \right\rangle = 0$$

for state with $l = 0$. Where $\vec{S}_e$ and $\vec{S}_P$ are the spin of electron and proton respectively.

3. (25%) (F in hyperfine structure) In class we demonstrated that the proton spin generates a vector potential $\vec{A}$ and a corresponding magnetic field $\vec{B}$. Such that the original Hamiltonian is perturbed by

$$H_1 = 2\mu_B \left(\vec{p} \cdot \vec{A} + \vec{s} \cdot \vec{B} \right),$$

and results in the hyperfine structure (see the lecture note on Mar.31). Show that F is a good quantum number, namely, $[H_1, F] = 0$, where $F = J + I = L + S + I$.

4. (25%) (Projection theorem) Assume that the matrix elements of a vector operator $\mathbf{V}$ in a subspace of definite total angular momentum are proportional to those of $\mathbf{J}$, show that

$$j(j+1) < jm|\mathbf{V}|jm' > = < jm|(\mathbf{V} \cdot \mathbf{J})\mathbf{J}|jm' >$$

Later, we will have a more general discussion on this and the so called Wigner-Eckart theorem will be introduced.