Now we return to QM, we first try the operator

$$\hat{M}_0 = \frac{1}{2}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - z\frac{\vec{r}}{r}$$

with $\hat{H} = \frac{\vec{p}^{\,2}}{2} - \frac{z}{r}$

one obtains

$$[\hat{M}_0, \hat{H}] = 0, \qquad \hat{M}_0\cdot\hat{L} = 0, \quad \text{and } \hat{M}_0\cdot\hat{M}_0 = 2\hat{H}(\hat{L}^2+1) + z^2$$

The derivation is left as a HW.

Rescaling as in CM.

$\hat{M} = \sqrt{\frac{1}{-2E}}\,\hat{M}_0$, it satisfies the following rules:

$$[\hat{M}_i, \hat{M}_j] = i\epsilon_{ijk}\hat{L}_k$$
$$[\hat{M}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{M}_k$$

together with the usual relations for angular momentum:

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$$

This algebra suggests that $\hat{M}$ can be viewed as an additional angular momentum

$\sim$ 6 angular momentums: $\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{M}_x, \hat{M}_y, \hat{M}_z$

$\| $

$C_2^4 \rightsquigarrow$ the angular momentum in 4D spatial dimension.

$$\hat{L}_x = yp_z - zp_y, \quad \hat{L}_y = zp_x - xp_z, \quad \hat{L}_z = xp_y - yp_x$$

$$\Rightarrow \hat{M}_x = xp_4 - x_4p_x, \quad \hat{M}_y = yp_4 - x_4p_y, \quad \hat{M}_z = zp_4 - x_4p_z$$

In addition to $[x_i, p_j] = i\delta_{ij}$ $\quad i,j = 1,2,3$

we also require that $[x_4, p_4] = i$,

or more compactly $[x_i, p_j] = i\delta_{ij}$ $\quad i,j = 1,2,3,4$

It is convenient to define

$$\hat{I} = \frac{1}{2}(\hat{L}+\hat{M})$$

$$\hat{K} = \frac{1}{2}(\hat{L}-\hat{M})$$

& $[\hat{I}_i, \hat{I}_j] = \frac{1}{4}[\hat{L}_i+\hat{M}_i, \hat{L}_j+\hat{M}_j] = \frac{1}{4} i \epsilon_{ijk}[\hat{L}_k + \hat{L}_k + \hat{M}_k \overset{(-)^2}{+} \hat{M}_k]$
$= i\epsilon_{ijk}\hat{I}_k$

Similarly, $[\hat{K}_i, \hat{K}_j] = \frac{1}{4}[\hat{L}_i-\hat{M}_i, \hat{L}_j-\hat{M}_j] = \frac{1}{4} i \epsilon_{ijk}[2\hat{L}_k - 2\hat{M}_k] = i\epsilon_{ijk}\hat{K}_k$

Furthermore,

$$[\hat{I}_i, \hat{K}_j] = \frac{1}{4}[\hat{L}_i+\hat{M}_i, \hat{L}_j-\hat{M}_j] = \frac{1}{4} i\epsilon_{ijk}[\hat{L}_k - \hat{L}_k + \hat{M}_k - \hat{M}_k] = 0$$

since $\hat{L}$ and $\hat{M}$ each commute with $\hat{H}$

$$\Rightarrow [\hat{I}, \hat{H}] = 0, \quad [\hat{K}, \hat{H}] = 0$$

$\Rightarrow$ Thus the eigenstates of $\hat{H}$ are simultaneously eigenstates of $\hat{I}^2$ & $\hat{K}^2$ with eigenvalues $I(I+1)$, $K(K+1)$, with, $I, K = \underline{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots}$

Do you know why?

Moreover, since $\hat{L}\cdot\hat{M} = 0$

$$\hat{I}^2 = \frac{1}{4}(\hat{L}^2+\hat{M}^2+2\hat{L}\cdot\hat{M}) = \hat{K}^2 = \frac{1}{4}(\hat{L}^2+\hat{M}^2-2\hat{L}\cdot\hat{M})$$

$$\Rightarrow \hat{I}^2+\hat{K}^2 = 2\hat{I}^2 = \frac{1}{2}(\hat{L}^2+\hat{M}^2) = \frac{1}{2}\left[\hat{L}^2\left(1-\frac{\hat{H}}{E}\right) - \frac{\hat{H}}{E} - \frac{Z^2}{2E}\right]$$

$$\left(\text{since } M^2 = \frac{1}{-2E}M_0^2 = -\frac{1}{2E}\left(2\hat{H}(\hat{L}^2+1) + Z^2\right)\right)$$

Plugging in their eigenvalues, we have

$$\frac{1}{2}\left[L(L+1)\left(1-\frac{E}{E}\right) - \frac{E}{E} - \frac{Z^2}{2E}\right) = 2I(I+1)$$

$$4I(I+1) = -1 - \frac{Z^2}{2E}$$

or $$\frac{Z^2}{2E} = -(4I^2+4I+1) = -(1+2I)^2$$

$$\Rightarrow \quad E = -\frac{Z^2}{2(2I+1)^2}$$

since, $I = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$
one may define the principle quantum #, $n = 2I+1$
$n = 1, 2, 3 \cdots$

$$\Rightarrow \boxed{E_n = -\frac{Z^2}{2n^2}}$$

This definition is consistent with

$$n = l_{max} + 1.$$

We have recovered the Balmer formula.
Just as in the classical case, E depends on the eigenvalues of $\hat{L}^2 + \hat{M}^2$.

$O4$ symmetry $\simeq$ 6 rotational generators (3 L, 3 M) in 4-D Euclidian space.