Another approach for spin

$$Q = L K_{sz} \qquad \Rightarrow \quad \text{representation depdent}.$$

$$K_{sz} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} K_{sz}^\dagger = \begin{pmatrix} S_x \\ -S_y \\ S_z \end{pmatrix}.$$

eg. $S = \frac{1}{2}$ $\quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

∵ S_y is pure imaginary in the S_z representation.

$S = 1.$ $\quad S_x = \hbar\begin{pmatrix} 1 & 1 \\ 1 & \\ & 1 \end{pmatrix}, \quad S_y = \hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

Therefore $\theta \neq k_{sz}$, to fix it,

$$S_z = \hbar\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{pmatrix}$$

need

$$U_0 = U(\hat{y}, \pi) = \exp\left[\frac{-i\pi S_y}{\hbar}\right], \quad \text{flips the sign of the } x\text{- and } z\text{-components}$$

$$\Rightarrow \quad U_0 \begin{pmatrix} S_x \\ -S_y \\ S_z \end{pmatrix} U_0^\dagger = -\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

$$\Rightarrow \quad \theta = e^{\frac{-i\pi S_y}{\hbar}} k = k e^{\frac{-i\pi S_y}{\hbar}} \qquad (\because i S_y \text{ is real })$$

You should check by explicit calculation that this θ leads to

$$\theta |s,m\rangle = (i)^{2m} |s, -m\rangle \qquad \text{same as before}$$

Spatial + spin D.O.F.

$$\mathcal{E} = \text{span} \{|r, m\rangle\} \qquad \text{two time-reversal operators}$$

$$\theta_r = k_r \quad \& \quad \theta_s = k_{sz} U_0$$

$$\Rightarrow \quad \theta = k_{r s_z} e^{\frac{-i\pi S_y}{\hbar}}, \qquad \text{eg. spin} -\frac{1}{2}, \quad S_y = \frac{\hbar}{2}\sigma_y$$

$$e^{\frac{-i\pi S_y}{\hbar}} = e^{-\frac{i\pi}{2}\sigma_y} = \cos\left(\frac{\pi}{2}\right) \mathbb{1} - i\sigma_y \sin\left(\frac{\pi}{2}\right) = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

spinor WF

$$\begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix} \xrightarrow{\theta} \begin{pmatrix} -\psi_-^*(\vec{r}) \\ +\psi_+^*(\vec{r}) \end{pmatrix}$$

More generally

$$\psi_m(\vec{r}) \xrightarrow{\theta} \sum_{m'} d^s_{mm'}(\pi)\, \psi_{m'}^*(\vec{r})$$

for $|\vec{r_1}, m_{s_1}\rangle \otimes \cdots \otimes |\vec{r_n}, m_{s_n}\rangle$

$$e^{-i\frac{\pi S_y}{\hbar}} = e^{-i\frac{\pi S_{1y}}{\hbar}} \cdots e^{-i\frac{\pi S_{ny}}{\hbar}}$$

Kramers' degeneracy

$$\theta^2 = k e^{-i\frac{\pi S_y}{\hbar}} k e^{-i\frac{\pi S_y}{\hbar}} = e^{-\frac{2\pi i S_y}{\hbar}} \qquad (\because k^2 = 1)$$

$\Rightarrow$ a 2π rotation about the y-axis

for every boson, the 2π rotation $\rightarrow +1$

$\quad$ '' fermion $\qquad$ '' $\qquad \rightarrow -1$

$$\Rightarrow \quad \theta^2 = \begin{cases} +1 & \text{even \# of fermions} \\ -1 & \text{odd \#} \quad \text{''} \end{cases}$$

In time-reversal invariant system with an odd \# of fermions, the energy levels are always degenerate.

Suppose non degenerate $|\psi\rangle$ with $E \Rightarrow \theta|\psi\rangle = e^{i\alpha}|\psi\rangle$

then $\quad \theta^2|\psi\rangle = \theta e^{i\alpha}|\psi\rangle = e^{-i\alpha}\theta|\psi\rangle = |\psi\rangle$

$\longrightarrow$ contradicts when $\theta^2 = -1 \Rightarrow$ degenerate

$|\psi\rangle$ & $\theta|\psi\rangle$ are orthogonal, but with same E !!

In an external $\vec{E}$ field, odd \# of electrons $\rightarrow$

each energy level must be at least 2 fold degenerate.