Fine Structure

* relativistic correction

$$E = \sqrt{p^2c^2 + \underset{\text{rest mass}}{\underset{\sim}{m_0^2}}c^4} = m_0c^2\sqrt{1+\frac{p^2}{m_0^2c^2}}$$

$$\sim \underbrace{m_0c^2}_{\substack{\text{const.}\\ \text{rest mass}\\ \text{no physical consequences.}}} + \underbrace{\frac{p^2}{2m_0}}_{\substack{\shortparallel\\ T:\text{ usual}\\ \text{kinetic}\\ \text{energy}}} - \underbrace{\frac{1}{8}\frac{p^4}{m_0^3c^2}}_{\substack{\text{first}\\ \text{relativistic correction}}} + \cdots$$

$$\frac{-p^4}{8m_0^3c^2} = -\frac{1}{2}\frac{1}{m_0c^2}(T^2)$$

In NR QM, $H_0 = T+V$, $E_{nl}^0 = -\frac{Z^2}{2n^2}$

$$\Rightarrow H'_{rel} = -\frac{T^2}{2m_0c^2} = -\frac{\alpha^2}{2}T^2 \quad (\text{in atomic unit})$$

$$\Rightarrow \Delta E_{rel}^{(1)} = -\frac{\alpha^2}{2}\langle nl|T^2|nl\rangle$$

$$= -\frac{\alpha^2}{2}\langle nl|(H_0-V)^2|nl\rangle$$

$$= -\frac{\alpha^2}{2}\left[E_{nl}^2 - 2E_{nl}\langle nl|V|nl\rangle + Z^2\langle nl|\frac{1}{r^2}|nl\rangle\right]$$

By virial theorem $\langle V\rangle = 2E$

& $\langle \frac{1}{r^2}\rangle = \frac{Z^2}{n^3(l+\frac{1}{2})}$

$$\Rightarrow \Delta E_{rel}^{(1)} = -\frac{\alpha^2}{2}\left[-3\left(\frac{Z^2}{2n^2}\right)^2 + \frac{Z^4}{n^3(l+\frac{1}{2})}\right] = \frac{Z^4\alpha^2}{2n^4}\left[\frac{3}{4} - \frac{n}{l+\frac{1}{2}}\right]$$

* The O_4 symmetry $\Rightarrow$ degeneracy, $\frac{Z^2}{2n^2}$ (indep. of l)

$\Rightarrow$ Runge-Lenz vector : the const semi-major axis

Including the special relativity $\Rightarrow$ semi-major axis precesses

$\Rightarrow$ RL vector is no longer conserved, O_4 is broken

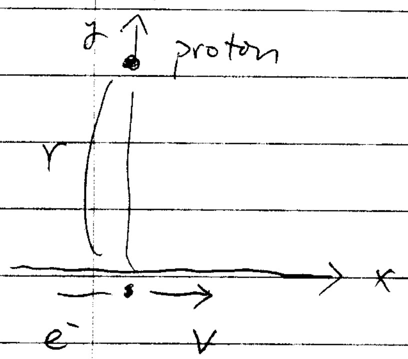
and the degeneracy is lifted

Spin - Orbital Interaction

If an electron moves with uniform velocity $\vec{V}$ in an electric field $\vec{E}$, from its point of view, it will also see an induced magnetic field.

$$\vec{B} = \gamma \frac{\vec{E}\times\vec{V}}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v^2}{c^2}$$

$$\underset{\text{if } \beta \ll 1}{\sim} \frac{\vec{E}\times\vec{V}}{c}$$



$$\vec{E}_y = -\frac{Z}{r^2} \quad \text{at proton's rest frame}$$

$$F^{\mu\nu}_{(p)} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & \frac{Z}{r^2} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & -\frac{Z}{r^2} & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}$$

In electron's frame, the proton moves backward

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \beta < 0.$$

$$B_z' = F^{21}_{(e)} = \Lambda^2_\alpha \Lambda^1_\beta F^{\alpha\beta}_{(p)} = \Lambda^1_\beta F^{2\beta}_{(p)} = \Lambda^1_0 F^{20}_{(p)} = +\gamma\beta \frac{Z}{r^2} \simeq \frac{vZ}{c}E$$

$\Rightarrow$ A magnetic field $B \sim \frac{v}{c}\frac{Z}{r^2}\hat{z}$ is seen by the electron.

$\because$ $L = mrv$ or $v = \frac{L}{mr}$ $\Rightarrow$ $\vec{B} = \frac{Z}{cmr^3}\vec{L}$

$$H'_{s.o.} = -\vec{\mu}\cdot\vec{B} = -g_s\mu_0 \vec{S}\cdot\vec{B} \qquad \left(\vec{\mu} = -g\frac{e}{2mc}\vec{S}, \quad \mu_0 = -\frac{|e|\hbar}{2m_ec} = -\frac{\alpha}{2}\right)$$

$$= \boxed{g_s \frac{\alpha^2}{2}\frac{Z}{r^3}\vec{L}\cdot\vec{S}}$$

$$\vec{J} = \vec{L} + \vec{S} \quad \Rightarrow \quad \vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$$

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{Z^3}{n^3 \ell(\ell+\frac{1}{2})(\ell+1)} \quad \text{for } \ell \neq 0$$

$$\Delta E^{(1)}_{S.O.} = \frac{\alpha^2 Z}{2}\left\langle\frac{1}{r^3}\right\rangle \frac{j(j+1) - \ell(\ell+1) - s(s+1)}{2}$$

for $\ell = 0$, consider the finite size of nucleus. $\Delta E^{(1)}_{10} = 0$.

$\Rightarrow$ Thomas precession. an extra factor $\frac{1}{2}$

$$\Delta E^{(1)}_{s.o} = \frac{\alpha^2 Z^4}{4} \frac{j(j+1) - \ell(\ell+1) - s(s+1)}{n^3 \, \ell(\ell+\frac{1}{2})(\ell+1)}$$

$\Rightarrow$ fine structure splitting between 2 states of the same n, $\ell(\neq 0)$, and s but different j.

e.g. $2^2_{P_{\frac{3}{2}}}$ and $2^2_{P_{\frac{1}{2}}}$

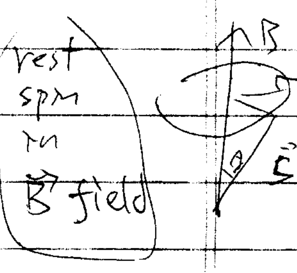
$$\delta E = \frac{Z^4\alpha^2}{4} \left\{ \frac{\left(\frac{3}{2}\frac{5}{2} - 1.2 - \frac{1}{2}\frac{3}{2}\right) - \left(\frac{1}{2}.\frac{3}{2} - 1.2 - \frac{1}{2}.\frac{3}{2}\right)}{8 \, . \, 1 \, . \, \frac{3}{2} \, . \, 2} \right\} = \frac{Z^4\alpha^2}{32}$$

agrees with Exp.

A handwaving argument to see the $\frac{1}{2}$ factor.

Thomas precession (Wigner rotation) : a pure relativistical kinematic effect, which also applies to the spinning sattelite moving around the Ear[th]

$H = -\vec{\mu}\cdot\vec{B}$, this is related to the Larmor frequency.

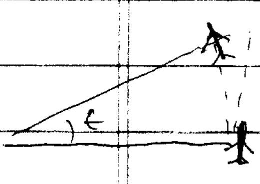


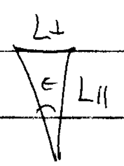
$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\mu}\times\vec{B} = \mu B \sin\theta$$
$$= \frac{\omega_L dt}{dt} L \sin\theta = \omega_L L \sin\theta$$
$$\Rightarrow \omega_L = \frac{\mu B}{L}$$

$\left(= \frac{eV}{mc^2} E\right.$ in Coulomb potential

$\omega_L = \omega_0$ (the circulating angular frequency)

~~However in the accellerating frame~~





$\epsilon = \frac{2\pi}{N} \simeq \frac{L_\perp}{L_\parallel}$, $N\epsilon = 2\pi$

After N steps, the plane completes a cirle.

As seen in plane's perspect $L_\parallel \to \frac{1}{\gamma} L_\parallel$ or $\epsilon_A \simeq \frac{L_\perp}{\frac{1}{\gamma}L_\parallel} = \gamma\epsilon$

For every step, to the rest frame, the plane miss ~~form an anglee~~

$\delta\epsilon \sim -(\gamma - 1)\epsilon$

$N\delta\epsilon \sim -\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)2\pi \sim -\frac{\beta^2}{2}(2\pi)$ $\Rightarrow$ Thomas precession

$$\boxed{\omega_T = -\frac{\beta^2}{2}\omega_0} = \boxed{-\frac{\beta^2}{2}\omega_L}$$

There fore, in the rest frame, the Larmor frequency becomes.

$$\omega_L \Rightarrow \omega_L - \tfrac{1}{2}\omega_L = \tfrac{1}{2}\omega_L \quad (\Leftrightarrow \vec{\mu}\cdot\vec{B})$$

see A.J.P 40, 1772 (1972)
A.J.P. 72, 943 (2004)
and Jackson, classical E&M.

$$\Delta E^{(1)}_{rel} + \Delta E^{(1)}_{s.o} = \frac{Z^4\alpha^2}{4n^4}\left[\frac{-2n}{\ell+\frac{1}{2}} + \frac{3}{2} + \frac{n[\,j(j+1) - \ell(\ell+1) - \frac{3}{4}\,]}{\ell(\ell+\frac{1}{2})(\ell+1)}\right]$$

$$= \frac{\alpha^2 Z^4}{4n^4}\left[\frac{3}{2} - \frac{2n}{j+\frac{1}{2}}\right]$$

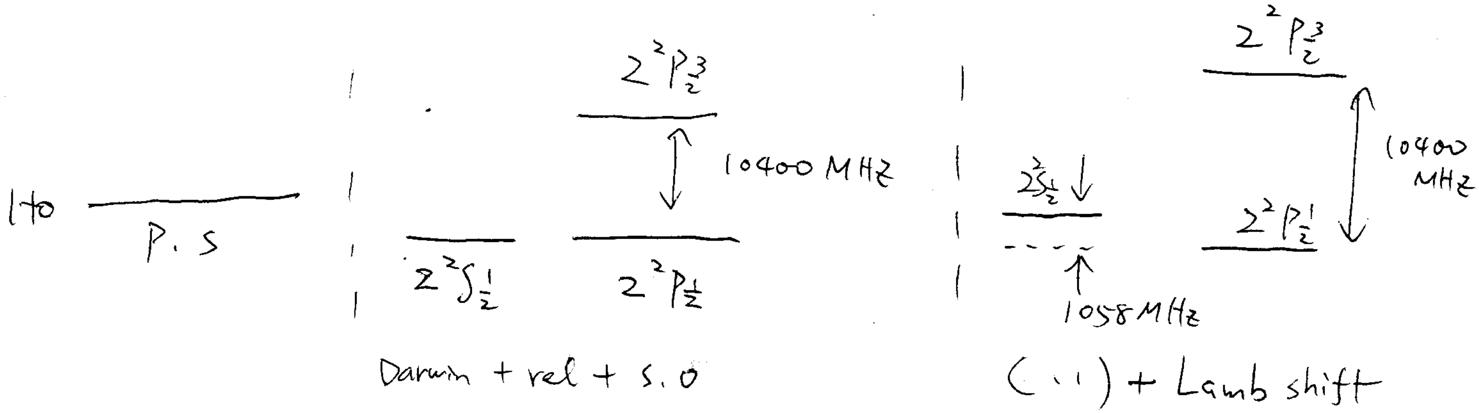
☆ However, there is a so called Darwin term (for s-state only)

$\Rightarrow$ shift the s-state energy

$\Rightarrow$ $^2P_{\frac{1}{2}}$ and $^2S_{\frac{1}{2}}$ of the same n are degenerate again!

☆ Moreover, the $S_{\frac{1}{2}} - P_{\frac{1}{2}}$ degeneracy is removed by the Lamb shift.

$= 1058$ MHZ, QED effect, after WW II



Hyper fine structure

Proton magnetic moment μ_p interacts with the magnetic field $\vec{B}_e$ generated by the electron's spin and orbital motion

$$H'_{hfs.} = -\vec{\mu_p} \cdot \vec{B_e}$$

$$\mu_p = \underbrace{g_p}_{\sim 5.58} \frac{e}{2m_p c} \vec{S}_p = g_p \frac{|e|\hbar}{2m_e c} \frac{m_e}{m_p} I = g_p \frac{\alpha}{2} \frac{m_e}{m_p} I$$

can be estimated by quark model

$m_p \sim 2000\, m_e$ a 3 order smaller correction than the fine structure.